



Lattice paths and generalized cluster complexes

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Abstract

In this paper we propose a variant of the generalized Schröder paths and generalized Delannoy paths by giving a restriction on the positions of certain steps. This generalization turns out to be reasonable, as attested by the connection with the faces of generalized cluster complexes of types *A* and *B*. As a result, we derive Krattenthaler's *F*-triangles for these two types by a combinatorial approach in terms of lattice paths. © 2008 Elsevier Inc. All rights reserved.

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1. Introduction

In their pioneering work [4,14], R. Simion et al. related the *f*-vector and *h*-vector of simplicial associahedra of types *A* and *B*, constructed in terms of polygon-dissections, to the enumeration of Schröder paths and Delannoy paths. We follow their footsteps to explore the explicit connection between the two structures in the context of '*s*-generalizations' of polygon-dissections and lattice paths.

1.1. Lattices paths and cluster complexes

Let us recall some classical facts about lattice paths. The *large Schröder numbers* $\{r_n\}_{n \geq 0} = \{1, 2, 6, 22, 90, 394, \dots\}$ [16, A006318] count the number of paths, called the *Schröder paths*

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of length n , in the plane $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$ to (n, n) using *north* steps $(0, 1)$, *east* steps $(1, 0)$, and *diagonal* steps $(1, 1)$ that never pass below the line $y = x$. The *small Schröder numbers* $\{s_n\}_{n \geq 0} = \{1, 1, 3, 11, 45, 197, \dots\}$ [16, A001003], which satisfy that $s_n = \frac{1}{2}r_n$ for $n \geq 1$, turn out to be the number of Schröder paths of length n with no diagonal steps on the line $y = x$. We refer the readers to [17, Exercise 6.39] for more information about Schröder numbers. The *central Delannoy numbers* $\{1, 3, 13, 63, 321, 1683, \dots\}$ [16, A001850] count the number of paths, called the *Delannoy paths* of length n , from $(0, 0)$ to (n, n) using the same set of steps without additional restrictions. Sulanke [18] collected a number of configurations that are counted by the central Delannoy numbers. See also [2] for a historical note on Delannoy numbers. A *Catalan path* (respectively *binomial path*) is simply a Schröder path (respectively Delannoy path) that contains no diagonal steps. Catalan paths of length n are enumerated by the Catalan numbers $c_n = \frac{1}{n+1} \binom{2n}{n}$ [16, A000108], and binomial paths of length n are enumerated by the central binomial numbers $b_n = \binom{2n}{n}$ [16, A000984].

Fomin and Zelevinsky [10] introduced a simplicial complex $\Delta(\Phi)$, called the *cluster complex*, associated to a root system Φ , which can be realized by a combinatorial structure constructed in terms of polygon-dissections. Specifically, for the case $\Phi = A_{n-1}$, the cluster complex $\Delta(A_{n-1})$, also called the *associahedron*, can be realized as a simplicial complex whose vertices are the diagonals of a regular $(n+2)$ -gon P , whose k -faces are the dissections of P using k noncrossing diagonals, and whose facets are the triangulations of P . The result of [4, Proposition 2.7] is equivalent to the observation that the k -face number of $\Delta(A_{n-1})$ is the number of Schröder paths of length n with $n-1-k$ diagonal steps, none of which is on the line $y = x$. In particular, the number of all faces is the small Schröder number s_n , and the number of facets is the Catalan number c_n . For the case $\Phi = B_n$, the cluster complex $\Delta(B_n)$, also called the *cyclohedron*, can be constructed similarly to the associahedron using centrally symmetric dissections of a regular $(2n+2)$ -gon. We refer the readers to [14, Section 2] for an explicit construction. In [14, Proposition 2] Simion observed that the k -face number of $\Delta(B_n)$ is the number of Delannoy paths of length n with $n-k$ diagonal steps.

1.2. The paths in connection to generalized cluster complexes

For a positive integer s , Fomin and Reading [9] defined a simplicial complex $\Delta^s(\Phi)$, called the *generalized cluster complex*, associated to a root system Φ and the integer s , which specializes at $s = 1$ to the cluster complex in [10]. The main purpose of this paper is to propose generalizations of Schröder paths and Delannoy paths that agree with the enumeration of the faces of $\Delta^s(\Phi)$ in types A and B , respectively.

Fixing s, r ($1 \leq r \leq s$), a *small (s, r) -Schröder path* of length n is a path π from $(0, 0)$ to (n, sn) using east steps, north steps, and diagonal steps that satisfies the following two conditions:

- (C1) the path π never passes below the line $y = sx$, and
- (C2) the diagonal steps of π are only allowed to go from the line $y = sj + r - 1$ to the line $y = sj + r$, for some j , but are not allowed to touch the line $y = sx$.

For example, the set of small $(2, 2)$ -Schröder paths of length 3 is shown in Fig. 1. Let $\mathcal{A}_n^{(s,r)}$ denote the set of such paths. A *block* of a path $\pi \in \mathcal{A}_n^{(s,r)}$ is a section beginning with a north step whose starting point is on the line $y = sx$ and ending with the first east step that returns to the

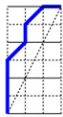
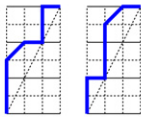
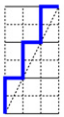
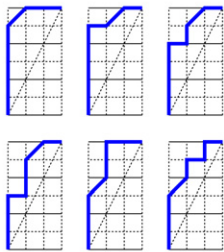
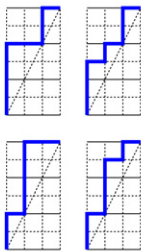
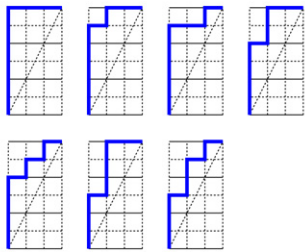
$d+b \backslash b$	1	2	3
3			
2			
1			

Fig. 1. The sets $\mathcal{A}_3^{(2,2)}(d, b)$ of small $(2, 2)$ -Schröder paths.

line $y = sx$ afterwards. Let $\mathcal{A}_n^{(s,r)}(d, b) \subseteq \mathcal{A}_n^{(s,r)}$ denote the set of paths with d diagonal steps and b blocks.

In the same spirit, an (s, r) -Delannoy path π of length n is a path from $(0, 0)$ to (n, sn) using the same set of steps that satisfies the following condition:

- (D) the diagonal steps of π are only allowed to go from the line $y = sj + r - 1$ to the line $y = sj + r$, for some j .

New results that we obtain are that the number of (s, r) -Schröder paths (respectively (s, r) -Delannoy paths) is independent of r (Theorems 2.4 and 4.1) and these paths are intimately related to faces of the generalized cluster complex. We shall establish bijections between the (s, r) -Schröder paths (respectively (s, r) -Delannoy paths) and the faces of $\Delta^s(\Phi)$ in type A (respectively B).

For the precise definition of $\Delta^s(\Phi)$, we refer the readers to [9,10]. Let P be a regular polygon with $sn + 2$ vertices. Consider the set of dissections of P into $(sj + 2)$ -gons ($1 \leq j \leq n - 1$)

by noncrossing diagonals. Such dissections are called *s-divisible*. For convenience, a diagonal in an *s*-divisible dissection is also called *s-divisible*. By Fomin and Reading's realization [9, Section 5.1], the simplicial complex $\Delta^s(A_{n-1})$ can be constructed using *s*-divisible diagonals of P , where the k -faces are the *s*-divisible dissections using k noncrossing diagonals. For the case $\Phi = B_n$, the simplicial complex $\Delta^s(B_n)$ can be constructed in a similar way using centrally symmetric *s*-divisible dissections of a regular $(2sn + 2)$ -gon. We refer the readers to [9, Section 5.2] for an explicit construction. See also [19].

Recently, Krattenthaler obtained the *F-triangles* of $\Delta^s(\Phi)$ [13], a refined enumerative result for the faces of $\Delta^s(\Phi)$ by the numbers of positive 'colored' roots and negative simple roots, which extends Chapoton's work [6]. Let $\mathcal{F}_\Phi^s(k, t)$ denote the set of faces of $\Delta^s(\Phi)$ that consist of k positive roots and t negative simple roots. In particular, the *F-triangles* of types *A* and *B* are given by

$$|\mathcal{F}_{A_{n-1}}^s(k, t)| = \frac{t+1}{k+t+1} \binom{n-1}{k+t} \binom{sn+k-1}{k}, \quad (1)$$

$$|\mathcal{F}_{B_n}^s(k, t)| = \binom{n}{k+t} \binom{sn+k-1}{k}. \quad (2)$$

For the case $\Phi = A_{n-1}$, we shall establish a bijection between the set $\mathcal{F}_{A_{n-1}}^s(k, t)$ and the set $\mathcal{A}_n^{(s,r)}(n-1-k-t, t+1)$, for each r ($1 \leq r \leq s$) (Theorem 3.1). Fig. 2 shows the sets $\mathcal{F}_{A_2}^2(k, t)$ of faces of $\Delta^2(A_2)$, in one-to-one correspondence with the sets $\mathcal{A}_3^{(2,2)}(2-k-t, t+1)$ of small $(2, 2)$ -Schröder paths shown in Fig. 1. Note that the number of elements in each cell forms the *F-triangle* of $\Delta^2(A_2)$. For the case $\Phi = B_n$, we establish a bijection between the faces of $\Delta^s(B_n)$ and the (s, r) -Delannoy paths of length n , which is given in a 'pair-to-pair' fashion due to the way the diagonals associated to the negative simple roots are specified (Theorem 5.1). This bijection is 'rare' in the sense that it is a two-to-two correspondence. As a result, we derive Krattenthaler's expression for the *F-triangles* in these two types by a pure combinatorial approach in terms of lattice paths.

We remark that these bijective results give an explicit connection between the *f*-vector of $\Delta^s(\Phi)$, which is given by

$$f_i(\Delta^s(\Phi)) = \sum_{k+t=i} |\mathcal{F}_\Phi^s(k, t)|, \quad (3)$$

for $0 \leq i \leq n$, in types *A* and *B* and the generalized Schröder paths and Delannoy paths, respectively. When specialized to $s = 1$, we obtain an explicit connection for the result by Simion et al. mentioned above.

We note that there are some generalized Schröder paths in the literature, for example, Song [15] considered a different version of generalized Schröder paths with the diagonal steps of all heights involved, i.e., without the restriction (C2).

1.3. Fuss–Narayana numbers

As shown in [9, Proposition 8.4], the number of facets of $\Delta^s(\Phi)$ is given by

$$\text{Cat}^{(s)}(\Phi) := \prod_{i=1}^n \frac{sh + e_i + 1}{e_i + 1}, \quad (4)$$

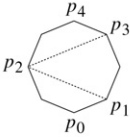
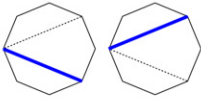

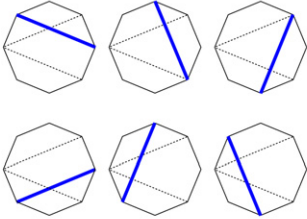
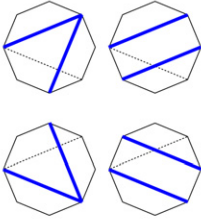
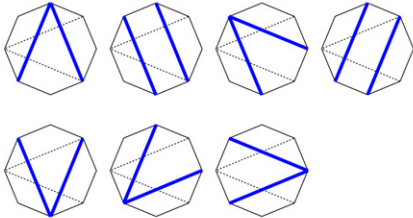
$k \backslash t$	0	1	2
0			
1			
2			

Fig. 2. The sets $\mathcal{F}_{A_2}^2(k, t)$ of faces of $\Delta^2(A_2)$.

known as the *Fuss–Catalan numbers* [1], where h is the Coxeter number and e_1, \dots, e_n are the exponents of Φ . In particular, $\text{Cat}^{(s)}(A_{n-1}) = \frac{1}{sn+1} \binom{sn+n}{n}$ coincides with the number of paths from $(0, 0)$ to (n, sn) using north steps and east steps that never pass below the line $y = sx$. Such paths are called *s-Catalan paths* of length n . We refer the readers to [11] for a list of objects, compiled by Heubach, Li, and Mansour, which are counted by the generalized Catalan number $\frac{1}{sn+1} \binom{sn+n}{n}$. For the case $\Phi = B_n$, $\text{Cat}^{(s)}(B_n) = \binom{sn+n}{n}$ coincides with the number of paths from $(0, 0)$ to (n, sn) without additional restriction. We call such paths *s-binomial paths* of length n . A *peak* of a path is formed by a pair of consecutive north step and east step.

Another combinatorial object in connection with Coxeter groups is the poset $NC^{(s)}(\Phi)$ formed by the s -divisible noncrossing partitions, developed by Armstrong [1] as a generalization of the case $s = 1$ given by Bessis [3] and Brady [5]. In [1, Definition 3.5.4], the *Fuss–Narayana numbers* $\text{Nar}^{(s)}(\Phi, k)$ are defined to be the k th rank numbers of $NC^{(s)}(\Phi)$. In particular, $\text{Nar}^{(s)}(A_{n-1}, k) = \frac{1}{n} \binom{n}{k} \binom{sn}{n-k-1}$ and $\text{Nar}^{(s)}(B_n, k) = \binom{n}{k} \binom{sn}{n-k}$. When $s = 1$ these numbers are the Narayana numbers of types A and B , which count the number of Catalan paths of length n with $k + 1$ peaks (see, for example, [8] and consult [12] for a survey) and the number of binomial paths of length n with k peaks (see, for example, [14, Proposition 2]), respectively. As noticed in [14], these numbers coincide with the h -vectors of associahedron $\Delta(A_{n-1})$ and cyclohedron $\Delta(B_n)$, respectively. However, when $s \geq 2$ these numbers do not agree with an obvious peak enumeration for the s -Catalan paths and the s -binomial paths. For example, as derived

by Cigler in [7], the number of s -Catalan paths of length n with k peaks is $\frac{1}{n} \binom{n}{k} \binom{sn}{k-1}$. In this paper we come up with a statistics that agrees with the Fuss–Narayana enumeration (Theorems 2.6 and 4.3).

We organize this paper as follows. For the type A case, the enumeration of the small (s, r) -Schröder paths and the statistics for the s -Catalan paths are given in Section 2, while the bijective results between the faces of $\Delta^s(A_{n-1})$ and the small (s, r) -Schröder paths are given in Section 3. The enumerative and bijective results for type B case are given in Sections 4 and 5, respectively. For the rest of this paper, we use the notation δ for dissections (or faces), π, τ, μ, ω for lattice paths, and $\phi, \varphi, \psi, \sigma, \zeta, \xi$ for bijections.

2. The (s, r) -Schröder paths and the s -Catalan paths

In this section, we enumerate the small (s, r) -Schröder paths with respect to blocks and diagonal steps, and enumerate the s -Catalan paths with respect to high peaks (see the definition below) by the method of generating functions.

2.1. Enumeration of the small (s, r) -Schröder paths

Given s and n , let $\mathcal{C}_n^{(s)}$ denote the set of s -Catalan paths of length n . A peak of a path $\pi \in \mathcal{C}_n^{(s)}$ is said to be at height h if its east step is on the line $y = h$. In particular, the peaks whose east steps do not touch the line $y = sx$ (i.e., the end point of the east step is not on the line $y = sx$) are called *high peaks*. Let N and E denote a north step and an east step, respectively. On the $sn \times n$ grid $G_{s,n}$ in the plane $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$ to (n, sn) , the set $\{y = st + j: 1 \leq j \leq s\}$ of lines is called the t th *stripe* of $G_{s,n}$. We devise an elementary operation on the path π that switches the high peak of π from one height to another within a specific stripe, which leads to the fact that the number of (s, r) -Schröder paths is independent of r .

Lemma 2.1. For $\epsilon \in \{0, 1\}$, let $H_\epsilon(h) \subseteq \mathcal{C}_n^{(s)}$ denote the set of paths with ϵ high peaks at height h . Then for $0 \leq t \leq n-1$ and $1 \leq i < j \leq s$, there is a bijection $\Gamma_{(t;i,j)}$ between $H_\epsilon(st+i)$ and $H_\epsilon(st+j)$.

Proof. We remind the readers that a *block* of a path in $\mathcal{C}_n^{(s)}$ was defined in the paragraph after (C2). Given a $\pi \in H_\epsilon(st+i)$ with a single block, we factorize π as $\pi = \mu_1 \omega_i \mu_2 \omega_j \mu_3$, where ω_i (respectively ω_j) is the section of east steps that lies on the line $y = st+i$ (respectively $y = st+j$), μ_1 goes from the point $(0, 0)$ to the line $y = st+i$, μ_2 goes from the line $y = st+i$ to the line $y = st+j$, and μ_3 goes from the line $y = st+j$ to the point (n, sn) . In particular, for the special case $st+j = sn$, i.e., on the top line of $G_{s,n}$, we assume that ω_j ends at the point $(n-1, sn)$ (i.e., the final east step E is excluded) and $\mu_3 = E$. Note that π has a high peak at height $y = st+i$ if and only if ω_i is nonempty. Then $\Gamma_{(t;i,j)}(\pi)$ is defined to be the path $\Gamma_{(t;i,j)}(\pi) = \mu_1 \omega_j \mu_2 \omega_i \mu_3 \in H_\epsilon(st+j)$, i.e., obtained from π by interchanging ω_i and ω_j .

For the paths π with more than one block, one can apply the operation block by block. \square

With the composition $\Gamma_{(0;i,j)} \circ \Gamma_{(1;i,j)} \cdots \circ \Gamma_{(n-1;i,j)}$ of the bijections, we have the following immediate result.

Corollary 2.2. For $k \geq 0$, let $\mathcal{H}_k(r) \subseteq \mathcal{C}_n^{(s)}$ denote the set of paths with k high peaks at heights h with $h \equiv r \pmod{s}$. Then there is a bijection between $\mathcal{H}_k(i)$ and $\mathcal{H}_k(j)$, for $1 \leq i < j \leq s$.

Fixing an r ($1 \leq r \leq s$), given a $\pi \in \mathcal{A}_n^{(s,r)}$, one can convert π into an s -Catalan path π' by replacing each diagonal step by a peak NE. To distinguish the replaced ones from the others, such a peak is colored red. Since the diagonal steps of π are only allowed to reach lines of the form $y = sj + r$, for some j , but are not allowed to touch the line $y = sx$, the red peaks of π' are high peaks at heights $h \equiv r \pmod{s}$. By the map in Corollary 2.2, it is straightforward to prove the following bijective result.

Lemma 2.3. For $k \geq 0$, let $\mathcal{G}_k(r) \subseteq \mathcal{A}_n^{(s,r)}$ denote the set of paths with k diagonal steps. Then there is a bijection between $\mathcal{G}_k(i)$ and $\mathcal{G}_k(j)$, for $1 \leq i < j \leq s$.

We shall enumerate the small (s, r) -Schröder paths by the number of diagonal steps and the number of blocks.

Theorem 2.4. Let s, n be positive integers. For $0 \leq d \leq n-1$ and $1 \leq b+d \leq n$, the cardinality of the set $\mathcal{A}_n^{(s,r)}(d, b)$ of small (s, r) -Schröder paths of length n with d diagonal steps and b blocks is given by

$$|\mathcal{A}_n^{(s,r)}(d, b)| = \frac{b}{n} \binom{n}{d} \binom{sn + n - b - d - 1}{n - b - d},$$

independent of r ($1 \leq r \leq s$).

Let $a_{n,k}$ be the number of paths in $\mathcal{A}_n^{(s,r)}$ with k diagonal steps. Define the generating polynomial for paths with respect to the number of diagonal steps and length of the paths

$$Q_r = Q_r(w, z) = \sum_{n \geq 0} \sum_{k=0}^{n-1} a_{n,k} w^k z^n.$$

It follows from Lemma 2.3 that $Q_1 = \cdots = Q_s$. We denoted this common polynomial for short by Q . For example, when $s = 2$ we have $Q = 1 + z + z^2(3 + w) + z^3(12 + 8w + w^2) + z^4(55 + 55w + 15w^2 + w^3) + \cdots$. To determine $a_{n,k}$, we consider the following paths that are permitted to pass below the line $y = sx$. A large (s, r) -Schröder path of length n is a path π from $(0, 0)$ to (n, sn) using north steps, east steps, and diagonal steps that satisfies the following two conditions: (i) π neither touches nor passes below the line $y = s(x-1)$, (ii) the diagonal steps of π are only allowed to go from the line $y = sj + r - 1$ to the line $y = sj + r$, for some j . Let $\mathcal{R}_n^{(s,r)}$ denote the set of such paths. Note that, given a $\pi \in \mathcal{A}_n^{(s,r)}$ with a single block, if π is factorized into $\pi = \hat{N}\tau E$, where \hat{N} denotes a section of s consecutive north steps, then $\tau \in \mathcal{R}_{n-1}^{(s,r)}$ (i.e., τ is obtained from π with the initial s north steps and the final east step E removed). Let $b_{n,k}$ be the number of paths in $\mathcal{R}_n^{(s,r)}$ with k diagonal steps, and define

$$R_r = R_r(w, z) = \sum_{n \geq 0} \sum_{k=0}^n b_{n,k} w^k z^n.$$

Similarly to before, $R_1 = \cdots = R_s$. Let R denote this common polynomial. For example, when $s = 2$ we have $R = 1 + z(2 + w) + z^2(7 + 6w + w^2) + z^3(30 + 36w + 12w^2 + w^3) + \cdots$. We

observe that Q and R satisfy the following equations:

$$Q = 1 + zR + (zR)^2 + \cdots = \frac{1}{1 - zR}, \quad (5)$$

$$R - Q = wzRQ^s + zRQ(Q + \cdots + Q^{s-1}). \quad (6)$$

For Eq. (5), we enumerate the number of small (s, r) -Schröder paths of length n by factorizing a path into blocks, where the possibilities of each block are enumerated by zR . For Eq. (6), we enumerate the number of large (s, s) -Schröder paths $\tau \in \mathcal{R}_n^{(s,s)} - \mathcal{A}_n^{(s,s)}$, which touch or pass below the line $y = sx$, where w marks a diagonal step of τ . Regarding the first time τ returns to the line $y = sx$, we observe that τ either touches the line $y = sx$ by a diagonal step D or passes below the line $y = sx$ by an east step E . For the former case, τ has a factorization $\tau = \mu_1 N_1 \cdots \mu_{s-1} N_{s-1} \mu_s D \omega$, where N_i is the last north step before D that rises from the line $y = sx + i - 1$ to the line $y = sx + i$, for $1 \leq i \leq s - 1$, each μ_i is a small $(s, s - i + 1)$ -Schröder path of a certain length, for $1 \leq i \leq s$, and ω is a large (s, s) -Schröder path of a certain length. (Note that μ_i and ω are possibly empty.) This contributes the first term on the right-hand side of (6). For the latter case, suppose that E is at height $h \equiv t \pmod{s}$, for some t ($1 \leq t \leq s - 1$). Then τ has a factorization $\tau = \mu_1 N_1 \cdots \mu_t N_t \mu_{t+1} E N_{t+1} \cdots N_s \omega$, where N_i is the last north step before E that rises from the line $y = sx + i - 1$ to the line $y = sx + i$, for $1 \leq i \leq t$, μ_i is a small $(s, s - i + 1)$ -Schröder path, for $1 \leq i \leq t + 1$, $N_{t+1} \cdots N_s$ are consecutive north steps that go from the line $y = sx - s + t$ to the line $y = sx$, and ω is a large (s, s) -Schröder path. This contributes zRQ^{t+1} to the second term on the right-hand side of (6).

Proof of Theorem 2.4. Setting $T = zR$, it follows from (5) and (6) that

$$T = z \cdot \frac{1 + wT}{(1 - T)^s}. \quad (7)$$

We shall compute $[w^d z^n] \{T^b\}$. By Lagrange inversion, e.g., see [17, Corollary 5.4.3], with $H(z) = z^b$ and $G(z) = (1 + wz)(1 - z)^{-s}$, we have

$$[z^n] \{T^b\} = \frac{1}{n} [z^{n-1}] H'(z) G(z)^n = \frac{b}{n} [z^{n-1}] \{z^{b-1} (1 + wz)^n (1 - z)^{-sn}\},$$

and upon extracting $[w^d]$,

$$\begin{aligned} [w^d z^n] \{T^b\} &= \frac{b}{n} [w^d z^{n-b}] \{(1 + wz)^n (1 - z)^{-sn}\} = \frac{b}{n} \binom{n}{d} [z^{n-b-d}] \{(1 - z)^{-sn}\} \\ &= \frac{b}{n} \binom{n}{d} \binom{sn + n - b - d - 1}{n - b - d}, \end{aligned}$$

as required. \square

Making use of (7) and computing $[w^d z^n] Q$, we have the following result, which coincides with the f -vector of $\Delta^s(A_{n-1})$ (cf. [9, Theorem 8.5]).

Corollary 2.5. For $0 \leq d \leq n - 1$, the number of small (s, r) -Schröder paths of length n with d diagonal steps is given by

$$\frac{1}{n} \binom{n}{d} \binom{sn + n - d}{n - d - 1},$$

independent of r ($1 \leq r \leq s$).

2.2. Enumeration of the s -Catalan paths

In the following, we enumerate the s -Catalan paths with respect to high peaks, which is a statistics that agrees with the Fuss–Narayana enumeration.

Theorem 2.6. *Let s, n be positive integers. For $0 \leq k \leq n-1$, the number of s -Catalan paths of length n with k high peaks at heights h such that $h \equiv r \pmod{s}$ is given by*

$$\frac{1}{n} \binom{n}{k} \binom{sn}{n-k-1},$$

independent of r ($1 \leq r \leq s$).

Fixing an r ($1 \leq r \leq s$), let $c_{n,k}$ be the number of paths in $\mathcal{C}_n^{(s)}$ with k high peaks at heights $h \equiv r \pmod{s}$. Define the generating polynomial for s -Catalan paths with respect to the number of high peaks and length of the paths

$$C_r = C_r(w, z) = \sum_{n \geq 0} \sum_{k=0}^{n-1} c_{n,k} w^k z^n.$$

It follows from Corollary 2.2 that $C_1 = \cdots = C_s$. Let C denote this common polynomial. For example, when $s = 2$ we have $C = 1 + z + z^2(2 + w) + z^3(5 + 6w + w^2) + z^4(14 + 28w + 12w^2 + w^3) + \cdots$. To determine $c_{n,k}$, we consider the following paths. An *escalated s -Catalan path* of length n is a path from $(0, 0)$ to (n, sn) using north steps and east steps that neither touches nor passes below the line $y = s(x - 1)$. Let $\mathcal{D}_n^{(s)}$ denote the set of such paths. Making use of Lagrange inversion, it is straightforward to derive that $|\mathcal{D}_n^{(s)}| = \frac{1}{s(n+1)-1} \binom{s(n+1)+n-1}{n+1}$. Note that this number coincides with the number of positive facets of $\Delta^s(A_n)$. Fixing an r ($1 \leq r \leq s$), let $d_{n,k}$ be the number of paths $\pi \in \mathcal{D}_n^{(s)}$ with k peaks at heights $h \equiv r \pmod{s}$. (Note that the east steps of the peaks might pass below the line $y = sx$.) Define

$$D_r = D_r(w, z) = \sum_{n \geq 0} \sum_{k=0}^n d_{n,k} w^k z^n.$$

From the proof of Lemma 2.1, we remark that the bijection in Corollary 2.2 holds as well when it is restricted to the set of s -Catalan paths of length $n+1$ with a single block. Given such a path π , if π is factorized into $\pi = \widehat{N}\tau E$, where \widehat{N} denotes a section of s consecutive north steps, then $\tau \in \mathcal{D}_n^{(s)}$. Hence there is a bijective result, similar to the one in Corollary 2.2, for escalated s -Catalan paths. It follows that $D_1 = \cdots = D_s$. Let D denote this common polynomial. For example, when $s = 2$ we have $D = 1 + z(1 + w) + z^2(2 + 4w + w^2) + z^3(5 + 15w + 9w^2 + w^3) + \cdots$. We observe that C and D satisfy the following equations:

$$C = \frac{1}{1 - zD}, \tag{8}$$

$$D - C = wzDC^s + zDC(C + \cdots + C^{s-2}). \tag{9}$$

For Eq. (8), we enumerate the set $\mathcal{C}_n^{(s)}$ in a way similar to (5). For Eq. (9), we enumerate the number of escalated s -Catalan paths $\tau \in \mathcal{D}_n^{(s)} - \mathcal{C}_n^{(s)}$, which pass below the line $y = sx$, where w

marks a peak of τ at height $h \equiv s - 1 \pmod{s}$. Namely, let E be the first east step of τ that passes below the line $y = sx$. Suppose that E is at height $h \equiv t \pmod{s}$, for some t ($1 \leq t \leq s - 1$). Then τ has a factorization $\tau = \mu_1 N_1 \cdots \mu_t N_t \mu_{t+1} E N_{t+1} \cdots N_s \omega$, where N_i is the last north step before E that rises from the line $y = sx + i - 1$ to the line $y = sx + i$, for $1 \leq i \leq t$, each μ_i is an s -Catalan path of a certain length, for $1 \leq i \leq t + 1$, $N_{t+1} \cdots N_s$ are consecutive north steps that go from the line $y = sx - s + t$ to the line $y = sx$, and ω is an escalated s -Catalan path of a certain length. On the right-hand side of (9), the first term is contributed by the case $t = s - 1$, and the other term is by the cases $1 \leq t \leq s - 2$. Now we are able to prove Theorem 2.6.

Proof of Theorem 2.6. Setting $T = zD$, it follows from (8) and (9) that

$$T = z \cdot \frac{1 - T + wT}{(1 - T)^s}. \quad (10)$$

Making use of Lagrange inversion and the same method as in the proof of Theorem 2.4, one can check that $[w^k z^n]C = \frac{1}{n} \binom{n}{k} \binom{sn}{n-k-1}$, as required. \square

Making use of (10) and computing $[w^k z^n]T^b$, we have the following result.

Corollary 2.7. For $1 \leq b \leq n$ and $0 \leq k \leq n - b$, the number of s -Catalan paths of length n with b blocks and k high peaks at heights h such that $h \equiv r \pmod{s}$ is given by

$$\frac{b}{n} \binom{n}{k} \binom{sn - b - 1}{n - k - b},$$

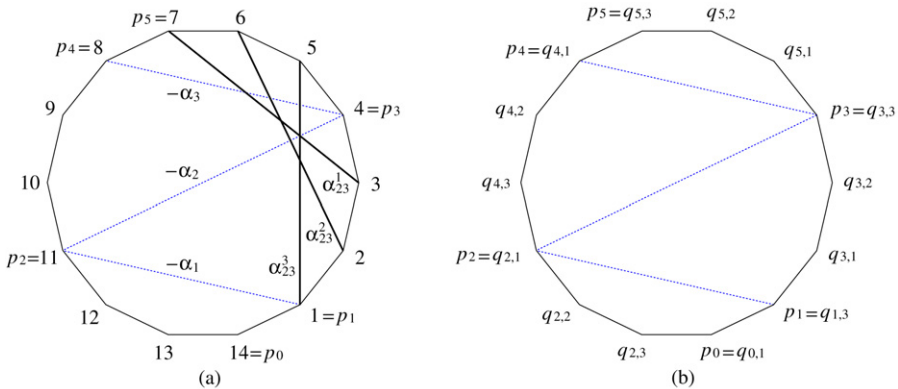
independent of r ($1 \leq r \leq s$).

3. The (s, r) -Schröder paths and the faces of $\Delta^s(A_{n-1})$

In this section, we establish the following refined bijection between the faces of $\Delta^s(A_{n-1})$ and the small (s, r) -Schröder paths of length n . Along with the enumerative result in Theorem 2.4, we derive Krattenthaler's expression for the F -triangles of type A combinatorially.

Theorem 3.1. Let s, n be positive integers. For $k, t \geq 0$ with $k + t \leq n - 1$ and for each r ($1 \leq r \leq s$), there is a bijection ψ between the sets $\mathcal{F}_{A_{n-1}}^s(k, t)$ and $\mathcal{A}_n^{(s,r)}(n - 1 - k - t, t + 1)$, where $\mathcal{F}_{A_{n-1}}^s(k, t)$ is the set of faces of $\Delta^s(A_{n-1})$ that consist of k positive roots and t negative simple roots, and $\mathcal{A}_n^{(s,r)}(n - 1 - k - t, t + 1)$ is the set of small (s, r) -Schröder paths of length n with $n - 1 - k - t$ diagonal steps and $t + 1$ blocks.

Let P be a regular polygon with $sn + 2$ vertices, labeled by $\{1, 2, \dots, sn + 2\}$ counter-clockwise. Following Fomin and Reading's realization [9, Section 5.1], we identify the faces of $\Delta^s(A_{n-1})$ with the s -divisible dissections of P . For the rest of this paper, whenever diagonals are mentioned we mean that they are s -divisible. The negative simple roots $\{-\alpha_1, \dots, -\alpha_{n-1}\}$ of A_{n-1} are identified with the following diagonals. For $1 \leq i \leq \frac{n}{2}$, the root $-\alpha_{2i-1}$ is identified with the diagonal connecting points $s(i - 1) + 1$ and $s(n - i) + 2$. For $1 \leq i \leq \frac{n-1}{2}$, the root $-\alpha_{2i}$ is identified with the diagonal connecting points $si + 1$ and $s(n - i) + 2$. These $n - 1$ diagonals form an s -snake of type A_{n-1} . For each positive root $\alpha_{ij} = \alpha_i + \cdots + \alpha_j$ ($1 \leq i \leq j \leq n - 1$) of A_{n-1} , there are exactly s diagonals intersecting the diagonals $-\alpha_i, \dots, -\alpha_j$ and no other diagonals in the s -snake. These diagonals are identified with positive 'colored' roots $\alpha_{ij}^1, \dots, \alpha_{ij}^s$

Fig. 3. The s -snake in type A_3 and the universal labeling.

accordingly. Fig. 3(a) shows the s -snake for $s = 3$ and $n = 4$, along with the diagonals identified with the positive roots α_{23}^1 , α_{23}^2 , and α_{23}^3 .

The bijection announced in Theorem 3.1 is constructed in several steps. An outline is as follows. First, we consider the positive facets of $\Delta^s(A_{n-1})$. Such a facet corresponds to an s -Catalan path with a single block. Next, we consider the positive k -faces ($k < n - 1$). For each r ($1 \leq r \leq s$), we devise a scheme to convert a k -face into a full dissection, and then obtain the corresponding (s, r) -Schröder path from the associated s -Catalan path. Finally, making use of the previous map as a building block, we deal with the faces that contain negative simple roots by dissecting P into subpolygons.

3.1. For the positive facets of $\Delta^s(A_{n-1})$

We shall establish a bijection between the full dissections of P that contain no diagonals in the s -snake and the s -Catalan paths with a single block. To describe the bijection, we devise two vertex-labeling schemes of P based on the position of the s -snake, with the benefits of explicit access to the roots of A_{n-1} . After setting up the schemes, the construction is downhill from there.

Let the vertices along the s -snake be p_1, p_2, \dots, p_n such that the endpoints of the diagonal $-\alpha_i$ are p_i and p_{i+1} , for $1 \leq i \leq n - 1$. Let the vertices next to p_1 and p_n be p_0 and p_{n+1} , respectively, i.e., $p_0 = sn + 2$ and $p_{n+1} = \lfloor \frac{sn+2}{2} \rfloor$. Note that for $2 \leq i \leq n + 1$ the points p_{i-2} and p_i are s points apart along the boundary of P . The $s - 1$ vertices between p_{i-2} and p_i along with p_i itself are labeled by $q_{(i,1)}, \dots, q_{(i,s)}$ in counterclockwise order along the boundary of P . An example for the case $s = 3$ and $n = 4$ is shown in Fig. 3(b). Let $S_i = \{q_{(i,1)}, \dots, q_{(i,s)}\}$ denote the i th sector. For convenience, let $p_0 = q_{(0,1)}$, $p_1 = q_{(1,3)}$ and let $S_0 = \{p_0\}$, $S_1 = \{p_1\}$. Each diagonal $D = q_{(a,b)}q_{(c,d)}$ is oriented from $q_{(a,b)}$ to $q_{(c,d)}$ if $a > c$, and $q_{(a,b)}$ is called the *initial point* of D . This scheme is called the *universal labeling*.

For the second scheme, we use the set $\{L_{(i,j)} : 2 \leq i \leq n + 1, 1 \leq j \leq s\}$ of labels. For $2 \leq i \leq n + 1$, the vertices of S_i are labeled by $L_{(i,1)}, \dots, L_{(i,s)}$ in the direction from p_0 to p_{n+1} along the boundary of P . (The vertex sets S_0, S_1 are not relabeled.) See Fig. 4 for an example. This scheme is called the *allocating labeling*, as the labels are used to indicate the heights of the east steps of a path. As one had noticed, the two labels for a specific vertex are related to each other by $q_{(i,j)} \leftrightarrow L_{(i,j')}$, where $j' = j$ if i is odd and $j' = s - j + 1$ otherwise.

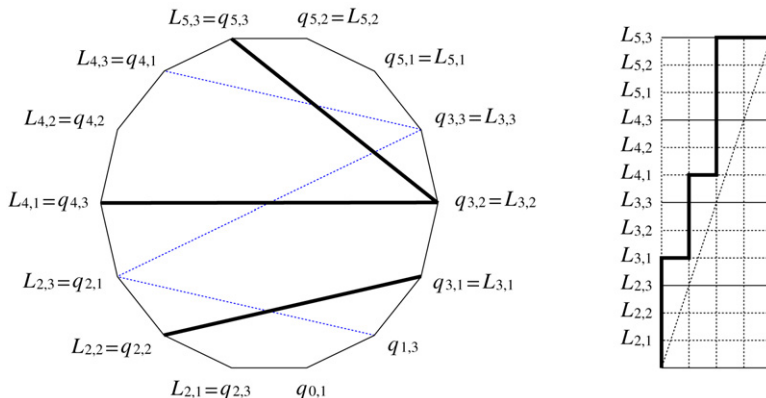


Fig. 4. A positive facet of $\Delta^3(A_3)$ and the corresponding 3-Catalan path.

On the grid $G_{s,n}$ from $(0, 0)$ to (n, sn) in the plane $\mathbb{Z} \times \mathbb{Z}$, the horizontal line $y = s(i - 2) + j$ is indexed by $L_{(i,j)}$, for $2 \leq i \leq n + 1$, $1 \leq j \leq s$. For convenience, we view $L_{(i,j)}$ as a height function $L_{(i,j)} = s(i - 2) + j$ of the grid and write $y = L_{(i,j)}$ for the height of the line $y = s(i - 2) + j$. Fig. 4 shows the two labeling schemes for a polygon of the case $s = 3$ and $n = 4$, along with the line-indexing for the grid $G_{3,4}$.

Now, we move on to the construction of the bijection.

Proposition 3.2. *There is a bijection ϕ between the set $\mathcal{F}_{A_{n-1}}^s(n - 1, 0)$ of positive facets of $\Delta^s(A_{n-1})$ and the set $\mathcal{C}_n^{(s)+}$ of s -Catalan paths of length n with one block.*

Proof. Given a $\delta \in \mathcal{F}_{A_{n-1}}^s(n - 1, 0)$, let the diagonals D_1, \dots, D_{n-1} of δ , whose initial points are labeled by $L_{(a_1,b_1)}, \dots, L_{(a_{n-1},b_{n-1})}$ respectively, be arranged in such a way that $(a_1, b_1) \leq \dots \leq (a_{n-1}, b_{n-1})$ in lexicographical order. On the grid $G_{s,n}$, we associate each D_k with an east step E_k at height $y = L_{(a_k,b_k)}$ in the k th column. Finally, we add an east step E_n at the top (i.e., on the line $y = sn$) of the n th column. Define $\phi(\delta)$ to be the unique path that contains the set of east steps $\{E_1, \dots, E_n\}$. We observe that for $1 \leq d \leq n - 1$ there are at most d diagonals in δ with such initial points $L_{(d+2,r)}$, for some r ($1 \leq r \leq s$), and then there are at most d east steps in $\{E_1, \dots, E_{n-1}\}$ at such heights $y = L_{(d+2,r)}$. Hence $\phi(\delta) \in \mathcal{C}_n^{(s)+}$ is well defined.

To find ϕ^{-1} , given a $\pi \in \mathcal{C}_n^{(s)+}$, let the k th east step E_k of π be at height $y = L_{(a_k,b_k)}$, for $1 \leq k \leq n - 1$. To recover the dissection $\phi^{-1}(\pi)$, we shall determine the diagonals associated to these steps by the following $(s + 2)$ -gon-removing process. By abuse of notation, given a point $v = L_{(i,j)}$ of the polygon P , we identify $u = L_{(i,j)} - d$ with the point u at distance d from v towards p_0 along the boundary of P .

For $k = 1$, if E_1, \dots, E_{n-1} are all at height $y = L_{(n+1,r)}$ for some r ($1 \leq r \leq s$), then $\phi^{-1}(\pi)$ is defined to be the unique dissection with $n - 1$ diagonals incident to the point $L_{(n+1,r)}$, otherwise we associate E_1 with the diagonal that connects the point $L_{(a_1,b_1)}$ and the point $L_{(a_1,b_1)} - (s + 1)$, in which case we remove the s vertices in between. Suppose that the diagonal associated to E_{k-1} has been determined. Let P' be the resulting polygon. If E_k, \dots, E_{n-1} are all at height $y = L_{(n+1,r)}$ for some r ($1 \leq r \leq s$), then we add $n - k$ diagonals incident to the point $L_{(n+1,r)}$ in P' and we are done, otherwise we associate E_k with the diagonal that connects the point $L_{(a_k,b_k)}$ and the point $L_{(a_k,b_k)} - (s + 1)$ along the boundary of P' and remove the s vertices in

between. One can check that none of these diagonals are in the s -snake, and hence $\phi^{-1}(\pi) \in \mathcal{F}_{A_{n-1}}^s(n-1, 0)$. \square

Example 3.3. Take $s = 3$ and $n = 4$. On the left of Fig. 4 there is a positive facet $\delta \in \Delta^3(A_3)$. The diagonals are oriented so that their initial points are $q_{(5,3)}$, $q_{(4,3)}$, and $q_{(3,1)}$, whose allocating labels are $L_{(5,3)}$, $L_{(4,1)}$, and $L_{(3,1)}$, respectively. On the right of Fig. 4 there is the corresponding 3-Catalan path, whose east steps E_1 , E_2 , and E_3 are at heights $y = L_{(3,1)}$, $L_{(4,1)}$, and $L_{(5,3)}$, respectively.

3.2. For the faces of $\Delta^s(A_{n-1})$ consisting of positive roots

We shall establish a bijection between the positive faces of $\Delta^s(A_{n-1})$ and the (s, r) -Schröder paths with a single block, i.e., the special case $t = 0$ of Theorem 3.1. Recall that in Lemma 2.1 we devise an operation on s -Catalan paths that switches the high peak from one height to another within a stripe. In the following, let $\Gamma_{(a,r)}$ denote the operation that switches the high peak of a path from height $y = L_{(a,r)}$ to height $y = L_{(a,s-r+1)}$, for some a, r ($2 \leq a \leq n+1$, $1 \leq r \leq s$).

Proposition 3.4. For $0 \leq k \leq n-1$ and for each r ($1 \leq r \leq s$), there is a bijection φ between the set $\mathcal{F}_{A_{n-1}}^s(k, 0)$ and the set $\mathcal{A}_n^{(s,r)}(n-1-k, 1)$.

Proof. The case $k = n-1$ is given in Proposition 3.2. Assume that $k < n-1$. Given a $\delta \in \mathcal{F}_{A_{n-1}}^s(k, 0)$, we turn δ into a full dissection $\delta' \in \mathcal{F}_{A_{n-1}}^s(n-1, 0)$ as follows. We need an alternative lexicographical order to determine the ‘least’ point (with respect to r) of a subpolygon: for $0 \leq i \leq n-1$, the $s-1$ vertices between p_i and p_{i+2} along with p_i itself are labeled by $M_{(i,1)}, \dots, M_{(i,s)}$ in the direction from p_0 to p_{n+1} along the boundary of P .

Note that δ dissects P into $(sj+2)$ -gons, for some j . Fixing an r , in each subpolygon P' with $sd+2$ vertices ($d \geq 2$), locate the vertex $M_{(i,j)}$ with the least i such that $M_{(i,j)}$ could be connected to a vertex of the form $q_{(a,r)}$, for some $a \geq 3$. Then we add $d-1$ diagonals incident to $M_{(i,j)}$. Note that when P is fully dissected there are $n-1-k$ additional diagonals, all of which have distinct initial points. We observe that such a dissection is always possible for each r ($1 \leq r \leq s$) due to the periodicity (of the second entries) of the universal labeling. These additional diagonals are colored *red*. By the map ϕ in Proposition 3.2, we construct the corresponding s -Catalan path $\phi(\delta')$ and color the peaks associated to red diagonals with a red east step. Note that the red peaks of $\phi(\delta')$ are high peaks at heights $y = L_{(a,r')}$, where $r' = r$ if a is odd and $r' = s-r+1$ otherwise. Then apply the operations $\Gamma_{(a,r')}$ on $\phi(\delta')$, for all even a , to switch the red peaks from height $y = L_{(a,s-r+1)}$ to height $y = L_{(a,r)}$. Finally, the requested path $\varphi(\delta)$ is obtained by converting the red peaks into diagonal steps. Since the $n-1-k$ diagonal steps reach heights $h \equiv r \pmod{s}$, $\varphi(\delta) \in \mathcal{A}_n^{(s,r)}(n-1-k, 1)$.

To find φ^{-1} , given a $\pi \in \mathcal{A}_n^{(s,r)}(n-1-k, 1)$, we turn π into an s -Catalan path π' by converting the $n-1-k$ diagonal steps into high peaks with a red east step. Then apply $\Gamma_{(a,r)}$ on π' , for all even a , so that the red peak at such a height $y = L_{(a,r)}$ is switched to height $y = L_{(a,s-r+1)}$. Let π'' denote the resulting path. By the map ϕ^{-1} in Proposition 3.2, we construct the corresponding dissection $\phi^{-1}(\pi'') \in \mathcal{F}_{A_{n-1}}^s(n-1, 0)$. For each red peak of π'' , say at height $y = L_{(a,r')}$, where $r' = r$ if a is odd and $r' = s-r+1$ otherwise, we associate it to a diagonal in $\phi^{-1}(\pi'')$, which is incident to the point $L_{(a,r')} = q_{(a,r)}$. If there is more than one

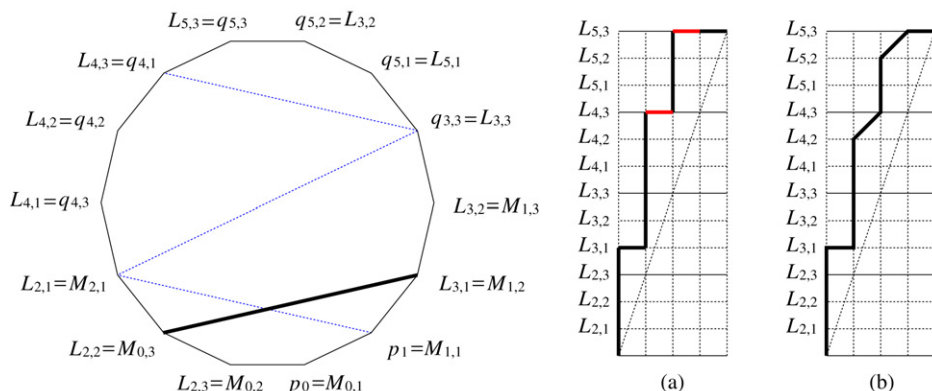


Fig. 5. A face of $\Delta^3(A_3)$ and the corresponding small $(3, 3)$ -Schröder path.

diagonal incident to $q_{(a,r)}$, then choose the one connecting to the point $M_{(i,j)}$ with the least i . Such a diagonal is colored red. Hence the requested dissection $\varphi^{-1}(\pi)$ is obtained from $\phi^{-1}(\pi'')$ with the red diagonals removed.

Since the argument works well for all r ($1 \leq r \leq s$), the assertion follows. \square

Example 3.5. Take $s = 3$ and $n = 4$. On the left of Fig. 5 there is a face $\delta \in \Delta^3(A_3)$ consisting of one positive root. We try to determine the corresponding small $(3, r)$ -Schröder path for $r = 3$ which contains 2 diagonal steps. In the upper subpolygon, $M_{(1,3)}$ is the ‘least’ point. We turn δ into a full dissection δ' by adding two diagonals incident to $M_{(1,3)}$, whose initial points are $q_{(4,3)}$ and $q_{(5,3)}$, respectively, as shown on the left of Fig. 4. By the map ϕ in Proposition 3.2, we construct the path $\phi(\delta')$ (see Example 3.3), as shown in Fig. 4. Applying the operation $\Gamma_{(4,1)}$ on $\phi(\delta')$, the peak at height $y = L_{(4,1)}$ is switched to $y = L_{(4,3)}$, shown in Fig. 5(a). Then the requested path $\varphi(\delta)$ is obtained by converting the peaks at heights $y = L_{(4,3)}$, $L_{(5,3)}$ into diagonal steps, as shown in Fig. 5(b).

3.3. For the faces of $\Delta^s(A_{n-1})$ consisting of positive roots and negative roots

Making use of the bijection in Proposition 3.4 as a building block, we are able to establish the bijection ψ of Theorem 3.1.

Proof of Theorem 3.1. The case $t = 0$ is given in Proposition 3.4. Assume that $t \geq 1$. Given a $\delta \in \mathcal{F}_{A_{n-1}}^s(k, t)$, let the negative simple roots of δ be $-\alpha_{j_1}, \dots, -\alpha_{j_t}$, where $1 \leq j_1 < \dots < j_t \leq n - 1$. These roots dissect P into $t + 1$ subpolygons P_1, \dots, P_{t+1} , where P_i and P_{i+1} are both adjacent to the diagonal $-\alpha_{j_i}$. Note that each P_i has $sn_i + 2$ vertices, where $n_i = j_i - j_{i-1}$ (we assume that $j_0 = 0$ and $j_{t+1} = n$). Let δ_i be the subdivision induced on P_i . Suppose that δ_i consists of k_i positive roots, so that $k_1 + \dots + k_{t+1} = k$. Fixing an r , by the map φ in Proposition 3.4, for each δ_i we construct the corresponding small (s, r) -Schröder path $\omega_i = \varphi(\delta_i)$, which is of length n_i with $n_i - 1 - k_i$ diagonal steps. Then the requested path $\psi(\delta)$ is defined by $\psi(\delta) = \omega_1 \cdots \omega_{t+1}$, i.e., by concatenation of the $t + 1$ blocks $\omega_1, \dots, \omega_{t+1}$. Note that $\psi(\delta)$ is of length $\sum_{i=1}^{t+1} n_i = n$ with $\sum_{i=1}^{t+1} (n_i - 1 - k_i) = n - 1 - k - t$ diagonal steps. Hence $\psi(\delta) \in \mathcal{A}_n^{(s,r)}(n - 1 - k - t, t + 1)$.

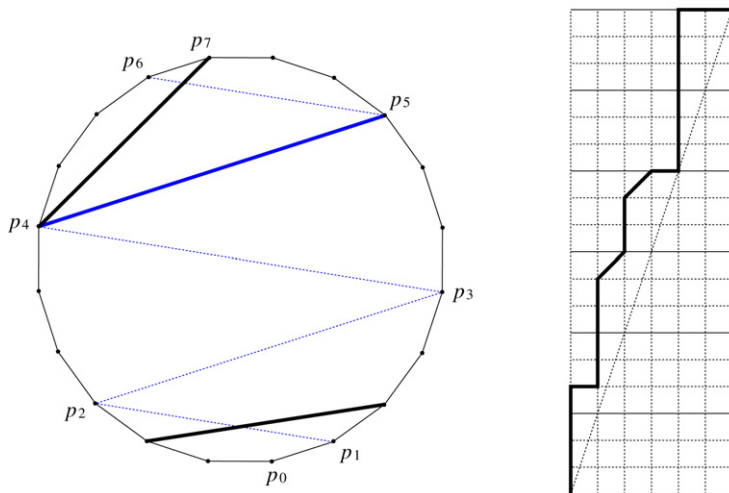


Fig. 6. A face of $\Delta^3(A_5)$ and the corresponding $(3, 3)$ -Schröder path.

To find ψ^{-1} , given a $\pi \in \mathcal{A}_n^{(s,r)}(n-1-k-t, t+1)$, we factorize π into $t+1$ blocks $\pi = \pi_1 \cdots \pi_{t+1}$. Suppose that π_i is of length n_i with d_i diagonal steps, so that $d_1 + \cdots + d_{t+1} = n-1-k-t$. For $1 \leq i \leq t$, let $j_i = n_1 + \cdots + n_i$. To recover the dissection $\psi^{-1}(\pi)$, we add the t negative simple roots $-\alpha_{j_1}, \dots, -\alpha_{j_t}$ in the s -snake, which dissect P into $t+1$ subpolygons P_1, \dots, P_{t+1} accordingly. Then we recover the subdivision $\varphi^{-1}(\pi_i)$ of P_i from π_i by the map φ^{-1} in Proposition 3.4, for $1 \leq i \leq t+1$. Since $\varphi^{-1}(\pi_i)$ consists of $n_i - 1 - d_i$ diagonals, $\psi^{-1}(\pi)$ contains $\sum_{i=1}^{t+1} (n_i - 1 - d_i) = k$ positive roots. Hence $\psi^{-1}(\pi) \in \mathcal{F}_{A_{n-1}}^s(k, t)$.

Since the argument works well for all r ($1 \leq r \leq s$), the proof is completed. \square

Example 3.6. Take $s = 3$ and $n = 6$. On the left of Fig. 6 there is a face $\delta \in \mathcal{F}_{A_5}^3(2, 1)$ consisting of two positive roots and the negative simple root $-\alpha_4 = p_4p_5$. We try to find the corresponding $(3, r)$ -Schröder path for $r = 3$. The polygon P is dissected by $-\alpha_4$ into subpolygons P_1 and P_2 , where P_1 (respectively P_2) contains the section $p_1p_2p_3p_4$ (respectively p_5p_6) of the s -snake. With respect to the sections of the s -snake, the dissection induced on P_1 is carried to a small $(3, 3)$ -Schröder path ω_1 by the map φ in Proposition 3.4 (see Example 3.5), shown in Fig. 5(b). The dissection induced on P_2 , consisting of p_4p_7 , is carried to a 3-Catalan path ω_2 . (One can make the subscripts of the point set $\{p_4, p_5, p_6, p_7\}$ ‘normalized’ to $\{p_0, p_1, p_2, p_3\}$.) The corresponding path $\psi(\delta) = \omega_1\omega_2$ is shown on the right of Fig. 6, where ω_1 is the section from $(0, 0)$ to $(4, 12)$, and ω_2 is the section from $(4, 12)$ to $(6, 18)$.

4. The (s, r) -Delannoy paths and the s -binomial paths

In this section, we enumerate the (s, r) -Delannoy paths, which are related to the F -triangle of type B , and enumerate the s -binomial paths with respect to peaks, which agree with the Fuss–Narayana enumeration of type B .

Given s, n , and r ($1 \leq r \leq s$), recall that the diagonal steps of the (s, r) -Delannoy paths $\pi \in \mathcal{B}_n^{(s,r)}$ are only allowed to reach the line $y = sj + r$, for some j . We enumerate the (s, r) -Delannoy paths π by the number of diagonal steps and the position at which π leaves the x -axis.

Theorem 4.1. Let s, n be positive integers. For $0 \leq d \leq n$ and $0 \leq d + t \leq n$, the cardinality of the set $\mathcal{B}_n^{(s,r)}(d, t)$ of (s, r) -Delannoy paths of length n with d diagonal steps, which visit the point $(t, 0)$ but not the point $(t + 1, 0)$, is given by

$$|\mathcal{B}_n^{(s,r)}(d, t)| = \binom{n}{d} \binom{sn + n - t - d - 1}{n - t - d},$$

independent of r ($1 \leq r \leq s$).

Proof. Given a path $\pi \in \mathcal{B}_n^{(s,r)}(d, t)$, there are d diagonal steps D_1, \dots, D_d at heights $h \equiv r \pmod{s}$ and $n - t - d$ east steps E_1, \dots, E_{n-t-d} above the x -axis. We observe that π is in one-to-one correspondence with a pair of sequences $0 \leq j_1 < \dots < j_d \leq n - 1$ and $1 \leq h_1 \leq \dots \leq h_{n-t-d} \leq sn$ such that D_i reaches height $sj_i + r$, for $1 \leq i \leq d$, and E_j is at height $h_j \in \{1, \dots, sn\}$, for $1 \leq j \leq n - t - d$. Hence $|\mathcal{B}_n^{(s,r)}(d, t)| = \binom{n}{d} \binom{sn + n - t - d - 1}{n - t - d}$.

Since the argument works well for all r ($1 \leq r \leq s$), the assertion follows. \square

By a summation over t , we have the following corollary. We remark that this coincides with the f -vector of $\Delta^s(B_n)$ (cf. [9, Theorem 8.5]).

Corollary 4.2. For $0 \leq d \leq n$, the number of (s, r) -Delannoy paths of length n with d diagonal steps is given by

$$\binom{n}{d} \binom{sn + n - d}{n - d},$$

independent of r ($1 \leq r \leq s$).

The following peak enumeration for the s -binomial paths agrees with the Fuss–Narayana enumeration of type B .

Theorem 4.3. Let s, n be positive integers. For $0 \leq k \leq n$, the number of s -binomial paths of length n with k peaks at heights h such that $h \equiv r \pmod{s}$ is given by

$$\binom{n}{k} \binom{sn}{n - k},$$

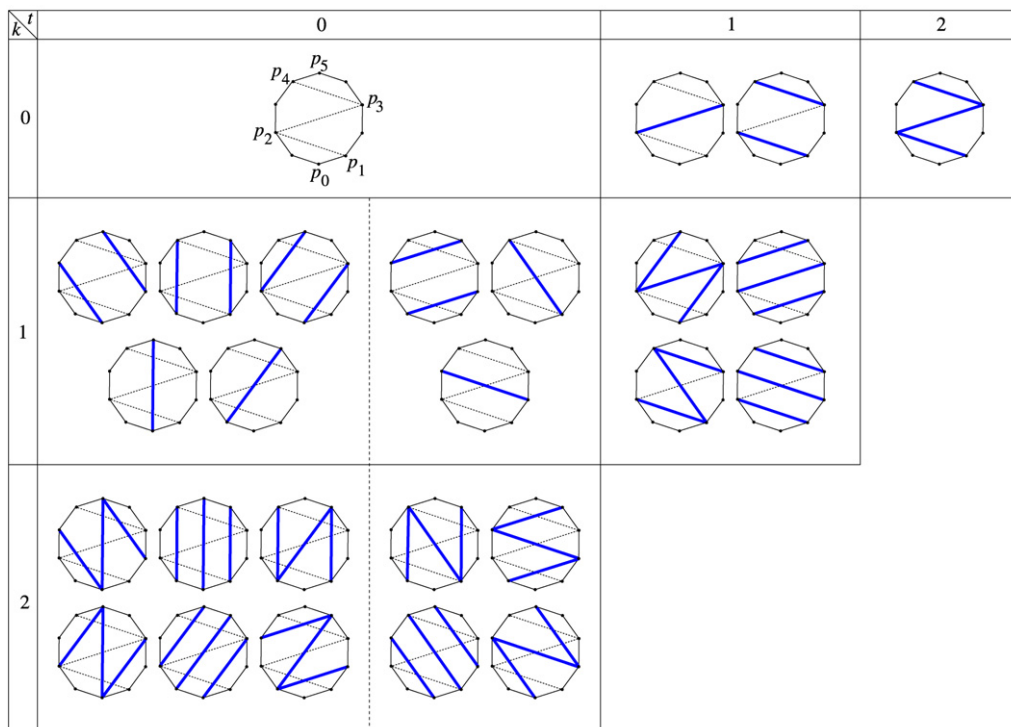
independent of r ($1 \leq r \leq s$).

Proof. Given such a path π , we associate the k peaks at heights $h \equiv r \pmod{s}$ with a k -tuple $0 \leq j_1 < \dots < j_k \leq n - 1$ of integers such that their heights are $y = sj_i + r$ accordingly. For the $n - k$ non-peak east steps, there is an $(n - k)$ -tuple $0 \leq h_1 \leq \dots \leq h_{n-k} \leq sn$ of integers for their heights. Since there is no other peak at heights $h \equiv r \pmod{s}$, $h_i \in \{0, 1, \dots, sn\} - \{st + r : 0 \leq t \leq n - 1, t \neq j_1, \dots, j_k\}$. Moreover, we observe that π is uniquely determined by such a pair of sequences (j_1, \dots, j_k) and (h_1, \dots, h_{n-k}) . Hence the number of the requested paths is $\binom{n}{k} \binom{sn}{n-k}$.

Since the argument works well for all r ($1 \leq r \leq s$), the assertion follows. \square

5. The (s, r) -Delannoy paths and the faces of $\Delta^s(B_n)$

In this section, we construct a bijection between the faces of $\Delta^s(B_n)$ and the (s, r) -Delannoy paths. Along with the enumerative result in Theorem 4.1, this leads to the F -triangle of $\Delta^s(B_n)$.

Fig. 7. The sets $\mathcal{F}_{B_2}^2(k, t)$ of faces of $\Delta^2(B_2)$.

5.1. An overview

First, we review the realization of $\Delta^s(B_n)$ given in [9, Section 5.2]. Let P be a regular polygon P with $2sn + 2$ vertices. A B -diagonal of P is either (i) a *diameter*, i.e., a diagonal that connects a pair of antipodal points, or (ii) a pair (D, D') of s -divisible diagonals such that D and D' are related by a half-turn rotation of P . The generalized cluster complex $\Delta^s(B_n)$ is a simplicial complex constructed on the set of B -diagonals of P , whose k -faces are the centrally symmetric dissections of P using k noncrossing B -diagonals. Its facets are the full dissections. To specify a face of $\Delta^s(B_n)$, we relate the roots of B_n to the s -snake of type A_{2n-1} . Let $\{p_1, \dots, p_{2n}\} \subseteq P$ be the s -snake type A_{2n-1} , with extension to p_0 and p_{2n+1} . Let $-\beta_1, \dots, -\beta_{2n-1}$ be the negative simple roots of A_{2n-1} . Recall that $-\beta_i$ is identified with the diagonal $p_i p_{i+1}$, for $1 \leq i \leq 2n-1$. Let $-\alpha_1, \dots, -\alpha_n$ be the negative simple roots of B_n . Then $-\alpha_i$ is encoded by the pair of diagonals corresponding to $-\beta_i$ and $-\beta_{2n-i}$, i.e., $-\alpha_i = (p_i p_{i+1}, p_{2n-i} p_{2n-i+1})$, for $1 \leq i \leq n-1$, while $-\alpha_n$ is encoded by the diameter corresponding to $-\beta_n$, i.e., $-\alpha_n = p_n p_{n+1}$. The positive roots in $\Delta^s(B_n)$ correspond to the B -diagonals that are not in the s -snake. We refer the readers to [9, Section 5.2] for an explicit correspondence. For illustration, the faces of $\Delta^2(B_2)$ are shown in Fig. 7.

Recall that $\mathcal{F}_{B_n}^s(k, t)$ is the set of faces of $\Delta^s(B_n)$ that consist of k positive roots and t negative simple roots, and that $B_n^{(s,r)}(d, t)$ is the set of (s, r) -Delannoy paths of length n with d diagonal steps, which visit the point $(t, 0)$ but not the point $(t+1, 0)$. We shall prove the following bijective result.

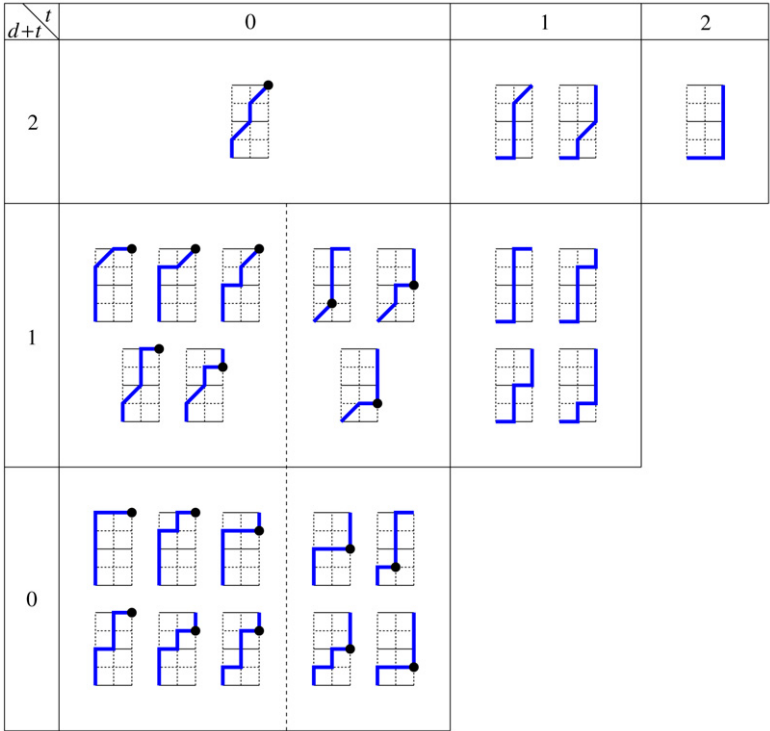


Fig. 8. The sets $\mathcal{B}_2^{(2,2)}(d, t)^+ \cup \mathcal{B}_2^{(2,1)}(d, t)^-$ of paths.

Theorem 5.1. *Let s, n be positive integers. For $k, t \geq 0$ with $k + t \leq n$ and for each r ($1 \leq r \leq \lceil \frac{s}{2} \rceil$), there is a bijection ξ between the set $2\mathcal{F}_{B_n}^s(k, t)$ and the set $\mathcal{B}_n^{(s,r)}(n - k - t, t) \cup \mathcal{B}_n^{(s, s-r+1)}(n - k - t, t)$, where $2\mathcal{F}_{B_n}^s(k, t)$ is a multiset consisting of two copies of each element in $\mathcal{F}_{B_n}^s(k, t)$.*

Given an (s, r) -Delannoy path π , a lattice point V in π is said to be at *level* ℓ if V is on the line $y = sx + \ell$. The *absolute minimum* of π is the point at the lowest level. If there is more than one point at the lowest level, choose the last one. The notion of absolute minimum plays an essential role in our method, as it links to the diameter of a full dissection of P .

The idea of the ‘pair-to-pair’ bijection $\xi : 2\mathcal{F} \rightarrow \mathcal{G} \cup \mathcal{H}$ is that both sets \mathcal{G} and \mathcal{H} are partitioned into two subsets $\mathcal{G} = \mathcal{G}^+ \cup \mathcal{G}^-$, $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$ (according to the positions of the absolute minima of the paths in \mathcal{G} and \mathcal{H}) and the bijection ξ is composed of two one-to-one correspondences $\xi_1 : \mathcal{F} \rightarrow \mathcal{G}^+ \cup \mathcal{H}^-$ and $\xi_2 : \mathcal{F} \rightarrow \mathcal{G}^- \cup \mathcal{H}^+$. Fig. 8 shows the sets $\mathcal{B}_2^{(2,2)}(d, t)^+ \cup \mathcal{B}_2^{(2,1)}(d, t)^-$ of paths, in one-to-one correspondence with the sets $\mathcal{F}_{B_2}^2(k, t)$ of faces of $\Delta^2(B_2)$ shown in Fig. 7. Note that the solid point in each path indicates the absolute minimum of the path.

The bijection announced in Theorem 5.1 is constructed in several steps as follows. First, we consider the positive facets of $\Delta^s(B_n)$. Such a facet corresponds to an s -binomial path that never visits the point $(1, 0)$. Next, we consider the positive k -faces ($k < n$). Making use of vertex-labeling schemes, for each r ($1 \leq r \leq s$), we devise a method to convert a k -face into a full

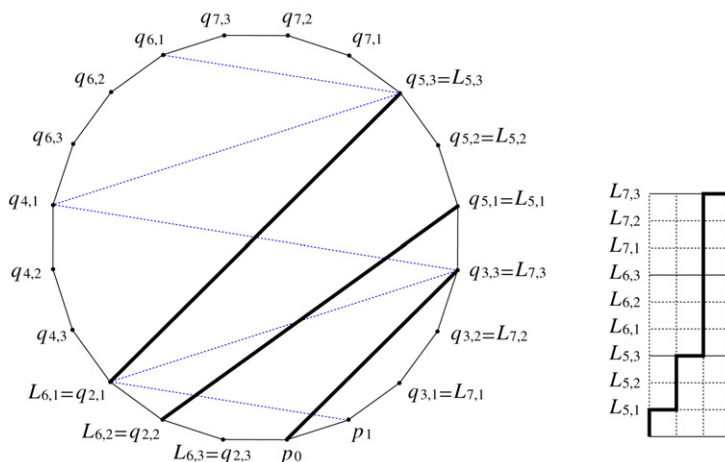


Fig. 9. A positive facet of $Q_3^{(3)}(2, 3)$ and the corresponding path in $X_3^{(3)}(2, 3)$.

dissection, and then obtain the corresponding (s, r) -Delannoy path or $(s, s - r + 1)$ -Delannoy path from the associated s -binomial path. Finally, we deal with the faces that contain negative simple roots by dissecting P into subpolygons.

5.2. For the positive facets of $\Delta^s(B_n)$

Our construction relies on two vertex-labeling schemes of P . The universal labeling $\{q_{(i,j)}: 2 \leq i \leq 2n+1, 1 \leq j \leq s\}$ of P is defined in exactly the same way as in Section 3, with the i th sector $S_i = \{q_{(i,1)}, \dots, q_{(i,s)}\}$, for $2 \leq i \leq 2n+1$, $S_0 = \{p_0\}$, and $S_1 = \{p_1\}$. See Fig. 9 for an example of the case $s = 3, n = 3$. The allocating labels $\{L_{(i,j)}: n+2 \leq i \leq 2n+1, 1 \leq j \leq s\}$ are dynamically assigned to the vertices of P , depending on the diameter of a facet. This scheme will be given in the proof of Proposition 5.2. As a height function for the grid $G_{s,n}$, we define $L_{(i,j)} = s(i - n - 2) + j$, as the line $y = s(i - n - 2) + j$ is indexed by $L_{(i,j)}$.

Now, we establish a bijection between the set $\mathcal{F}_{B_n}^s(n, 0)$ of positive facets of $\Delta^s(B_n)$ and the set $\mathcal{X}_n^{(s)}$ of s -binomial paths of length n that never visit the point $(1, 0)$. As a strengthening, this bijection is refined as follows. For $0 \leq b \leq n-1$ and $1 \leq r \leq s$, let $\mathcal{Q}_n^{(s)}(b, r) \subseteq \mathcal{F}_{B_n}^s(n, 0)$ be the set of positive facets that contain the diameter incident to the point $q_{(2n+1-b,r)}$. Given a path $\pi \in \mathcal{X}_n^{(s)}$, the *absolute minimum east step* (AMES) of π is the east step that reaches the absolute minimum. For $0 \leq b \leq n-1$ and $1 \leq r \leq s$, let $\mathcal{X}_n^{(s)}(b, r) \subseteq \mathcal{X}_n^{(s)}$ be the set of paths whose AMES are at height $y = L_{(2n+1-b,r)}$.

Proposition 5.2. *For $0 \leq b \leq n-1$ and $1 \leq r \leq s$, there is a bijection σ between the set $\mathcal{Q}_n^{(s)}(b, r)$ and the set $\mathcal{X}_n^{(s)}(b, r')$, where $r' = r$ if b is even and $r' = s - r + 1$ otherwise.*

Proof. Note that every full dissection of P contains a diameter, and that the dissection is completely determined by the subdivision on either side of the diameter. Fixing b, r and given a $\delta \in \mathcal{Q}_n^{(s)}(b, r)$, we specify the half subpolygon Q and the allocating labeling, depending on the position of the diameter as follows. There are two cases.

Case I. b is even.

Subcase (i) $r = s$, i.e., the diameter of δ is incident to p_{2n+1-b} . The vertex set of Q is the union of $X = S_{2n+1-b} \cup S_{2n-1-b} \cup \dots \cup S_{b+3}$ and $Y = S_{b+1} \cup S_b \cup \dots \cup S_0$. Note that Y contains the section from p_1 to p_{b+1} of the s -snake if $b \geq 2$. In X , for $0 \leq i \leq n - b - 1$, the vertices of $S_{2n+1-b-2i}$ are labeled by $L_{(2n+1-b-i,1)}, \dots, L_{(2n+1-b-i,s)}$ in counterclockwise order along the boundary of P , i.e., $q_{(2n+1-b-2i,j)} = L_{(2n+1-b-i,j)}$, for $1 \leq j \leq s$. If $b \geq 2$, then the vertices in Y are labeled as follows. For $0 \leq i \leq b - 1$, the vertices of S_{b+1-i} are labeled by $L_{(2n+1-i,1)}, \dots, L_{(2n+1-i,s)}$ in counterclockwise order along the boundary of P , i.e., $q_{(b+1-i,j)} = L_{(2n+1-i,j)}$, for $1 \leq j \leq s$.

Subcase (ii) $r < s$. Then δ contains the diameter $q_{(2n+1-b,r)}q_{(b+2,r+1)}$. Let Q be the half subpolygon containing p_0 . The vertex set $X = S_{2n+1-b} \cup S_{2n-1-b} \cup \dots \cup S_{b+3}$ is labeled in the same manner as above. (Note that $q_{(2n+1-b,k)} \notin Q$, for $r + 1 \leq k \leq s$.) If $b \geq 2$, then we assign the remaining allocating labels to the following subsets of Y . For $0 \leq i \leq b - 1$, let

$$T_{b+2-i} = \{q_{(b+2-i,j)} : r + 1 \leq j \leq s\} \cup \{q_{(b+1-i,j)} : 1 \leq j \leq r\},$$

and let these vertices be labeled by $L_{(2n+1-i,1)}, \dots, L_{(2n+1-i,s)}$ so that the second entries of the two labelings are consistent, i.e., $q_{(b+2-i,j)} = L_{(2n+1-i,j)}$, for $r + 1 \leq j \leq s$ and $q_{(b+1-i,j)} = L_{(2n+1-i,j)}$, for $1 \leq j \leq r$.

Case II. b is odd.

Subcase (i) $r = 1$, i.e., the diameter of δ is incident to p_{2n+1-b} . Then $Q = X \cup Y$, where $X = S_{2n+1-b} \cup S_{2n-1-b} \cup \dots \cup S_{b+3}$ and $Y = S_{b+1} \cup S_b \cup \dots \cup S_0$. In X , for $0 \leq i \leq n - b - 1$, the vertices of $S_{2n+1-b-2i}$ are labeled by $L_{(2n+1-b-i,1)}, \dots, L_{(2n+1-b-i,s)}$ in clockwise order along the boundary of P , i.e., $q_{(2n+1-b-2i,j)} = L_{(2n+1-b-i,s-j+1)}$, for $1 \leq j \leq s$. In Y , for $0 \leq i \leq b - 1$, the vertices of S_{b+1-i} are labeled by $L_{(2n+1-i,1)}, \dots, L_{(2n+1-i,s)}$ in clockwise order along the boundary of P , i.e., $q_{(b+1-i,j)} = L_{(2n+1-i,s-j+1)}$, for $1 \leq j \leq s$.

Subcase (ii) $r > 1$. Then δ contains the diameter $q_{(2n+1-b,r)}q_{(b+2,r-1)}$. Let Q be the half subpolygon containing p_0 . The vertex set $X = S_{2n+1-b} \cup S_{2n-1-b} \cup \dots \cup S_{b+3}$ is labeled in the same manner as above. (Note that $q_{(2n+1-b,k)} \notin Q$, for $1 \leq k \leq r - 1$.) For the set Y , we assign the remaining allocating labels to the following subsets. For $0 \leq i \leq b - 1$, let

$$T_{b+2-i} = \{q_{(b+2-i,j)} : 1 \leq j \leq r - 1\} \cup \{q_{(b+1-i,j)} : r \leq j \leq s\},$$

and let these vertices be labeled by $L_{(2n+1-i,1)}, \dots, L_{(2n+1-i,s)}$ so that the second entries of the two labelings are ‘reversely consistent,’ i.e., $q_{(b+2-i,j)} = L_{(2n+1-i,s-j+1)}$, for $1 \leq j \leq r - 1$ and $q_{(b+1-i,j)} = L_{(2n+1-i,s-j+1)}$, for $r \leq j \leq s$.

Under such an allocating scheme, we construct the bijection in a way similar to the one in Proposition 3.2. We arrange the diagonals D_1, \dots, D_n in Q , whose initial points are $L_{(a_1,b_1)}, \dots, L_{(a_n,b_n)}$, in such a way that $(a_1, b_1) \leq \dots \leq (a_n, b_n)$. On the grid $G_{s,n}$, we associate each diagonal D_k with an east step E_k at height $y = L_{(a_k,b_k)}$ in the k th column, for $1 \leq k \leq n$. Define $\sigma(\delta)$ to be the unique path that contains the set of east steps $\{E_1, \dots, E_n\}$. As the initial point $q_{(2n+1-b,r)}$ of the diameter is labeled by $L_{(2n+1-b,r')}$, where $r' = r$ if b is even and $r' = s - r + 1$ otherwise, one can check that the AMES of $\sigma(\delta)$ is located at height $y = L_{(2n+1-b,r')}$. Hence $\sigma(\delta) \in \mathcal{X}_n^{(s)}(b, r')$.

To find σ^{-1} , given a $\pi \in \mathcal{X}_n^{(s)}(b, r')$, let the k th east step E_k of π be at height $y = L_{(a_k,b_k)}$, for $1 \leq k \leq n$. Let $r = r'$ if b is even and $r = s - r' + 1$ otherwise. From the height $y = L_{(2n+1-b,r')}$ of the AMES of π , say E_d , we determine the diameter of $\sigma^{-1}(\pi)$, which is incident to the point

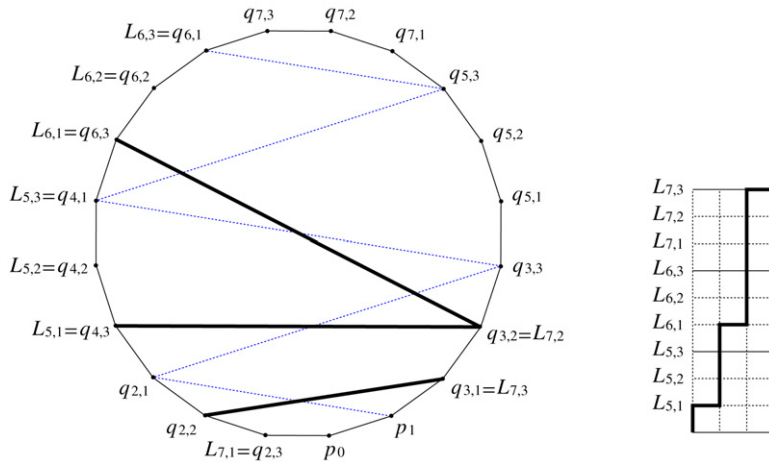


Fig. 10. A positive facet of $\mathcal{Q}_3^{(3)}(1, 3)$ and the corresponding path in $\mathcal{X}_3^{(3)}(1, 1)$.

$q_{(2n+1-b,r)}$. From this, the half subpolygon Q as well as the allocating labeling are determined, as defined above. To recover the diagonals in Q , we first determine the diagonals within Y , which are associated to E_{d+1}, \dots, E_n . For $d+1 \leq j \leq n$, we associate E_j to the diagonal that connects the point $L_{(a_j, b_j)}$ and the point $L_{(a_j, b_j)} - (s+1)$, and remove the s vertices in between. Then we determine the rest of the diagonals. For $1 \leq i \leq d-1$, we associate E_i to the diagonal that connects the point $L_{(a_i, b_i)}$ to the point $L_{(a_i, b_i)} - (s+1)$, and remove the s vertices in between. Then $\sigma^{-1}(\pi) \in \mathcal{Q}_n^{(s)}(b, r)$ is completely determined from Q . \square

Example 5.3. Take $s = 3$ and $n = 3$. On the left of Fig. 9 there is a facet $\delta \in \mathcal{Q}_3^{(3)}(2, 3)$, whose diameter is incident to $q_{(5,3)}$. Only the half subpolygon Q is shown, whose vertex set is the union of $X = S_5$ and $Y = S_3 \cup S_2 \cup \{p_0, p_1\}$, along with the allocating labeling. Note that the diagonals are oriented so that their initial points are $q_{(5,3)}$, $q_{(5,1)}$, and $q_{(3,3)}$, whose allocating labels are $L_{(5,3)}$, $L_{(5,1)}$ and $L_{(7,3)}$, respectively. The corresponding s -binomial path $\sigma(\delta)$ is shown on the right. Note that the second east step is the AMES, which is at height $y = L_{(5,3)}$. Hence $\sigma(\delta) \in \mathcal{X}_3^{(3)}(2, 3)$.

Example 5.4. On the left of Fig. 10 there is a facet $\delta \in \mathcal{Q}_3^{(3)}(1, 3)$, whose diameter is incident to $q_{(6,3)}$. Note that the allocating labels for S_6 and S_4 are $S_6 = \{L_{(6,3)}, L_{(6,2)}, L_{(6,1)}\}$ and $S_4 = \{L_{(5,3)}, L_{(5,2)}, L_{(5,1)}\}$. Moreover, the vertices of $T_3 = \{q_{(3,1)}, q_{(3,2)}, q_{(2,3)}\}$ are labeled by $L_{(7,3)}, L_{(7,2)}, L_{(7,1)}$, accordingly. The diagonals of Q are oriented so that their initial points are $q_{(6,3)}$, $q_{(4,3)}$, and $q_{(3,1)}$, whose allocating labels are $L_{(6,1)}$, $L_{(5,1)}$ and $L_{(7,3)}$, respectively. The corresponding s -binomial path $\sigma(\delta)$ is shown on the right. Note that the second east step is the AMES, which is at height $y = L_{(6,1)}$. Hence $\sigma(\delta) \in \mathcal{X}_3^{(3)}(1, 1)$.

By Proposition 5.2, we prove the special case $k = n, t = 0$ of Theorem 5.1.

Corollary 5.5. *There is a bijection between the set $\mathcal{F}_{B_n}^s(n, 0)$ and the set $\mathcal{X}_n^{(s)}$.*

5.3. For the faces of $\Delta^s(B_n)$ consisting of positive roots

Consider the set $\mathcal{B}_n^{(s,r)}(d, 0)$ of (s, r) -Delannoy paths with d diagonal steps, which never visit the point $(1, 0)$. Given a $\pi \in \mathcal{B}_n^{(s,r)}(d, 0)$, define a function ams on $\mathcal{B}_n^{(s,r)}(d, 0)$ by $\text{ams}(\pi) = b$ if the absolute minimum of π is at height $y = L_{(2n+1-b, j)}$, for some j ($1 \leq j \leq s$), which indicates the stripe to which the absolute minimum of π belongs. We partition $\mathcal{B}_n^{(s,r)}(d, 0)$ into two subsets $\mathcal{B}_n^{(s,r)}(d, 0)^+$ and $\mathcal{B}_n^{(s,r)}(d, 0)^-$, where

$$\mathcal{B}_n^{(s,r)}(d, 0)^+ = \{\pi \in \mathcal{B}_n^{(s,r)}(d, 0) : \text{ams}(\pi) \text{ is even}\}, \quad (11)$$

$$\mathcal{B}_n^{(s,r)}(d, 0)^- = \{\pi \in \mathcal{B}_n^{(s,r)}(d, 0) : \text{ams}(\pi) \text{ is odd}\}. \quad (12)$$

Note that $\mathcal{B}_n^{(s,r)}(d, 0)^- = \emptyset$ (i.e., $\mathcal{B}_n^{(s,r)}(d, 0)^+ = \mathcal{B}_n^{(s,r)}(d, 0)$) if $n = 1$. The following pair of bijections serve to establish the special case $k \leq n$, $t = 0$ of Theorem 5.1.

Proposition 5.6. *For $0 \leq k \leq n$ and for each r ($1 \leq r \leq s$), there is a bijection ζ_1 between the set $\mathcal{F}_{B_n}^s(k, 0)$ and the set $\mathcal{B}_n^{(s,r)}(n-k, 0)^+ \cup \mathcal{B}_n^{(s, s-r+1)}(n-k, 0)^-$, and there is a bijection ζ_2 between the set $\mathcal{F}_{B_n}^s(k, 0)$ and the set $\mathcal{B}_n^{(s,r)}(n-k, 0)^- \cup \mathcal{B}_n^{(s, s-r+1)}(n-k, 0)^+$.*

Proof. For $k = n$, note that $\mathcal{B}_n^{(s,r)}(0, 0) = \mathcal{B}_n^{(s, s-r+1)}(0, 0) = \mathcal{X}_n^{(s)}$ since the paths contain no diagonal steps. Hence $\zeta_1 = \zeta_2$ is the bijection given in Corollary 5.5. Assume that $k < n$. Recall the alternative lexicographical order $M_{(i,j)}$ defined in the proof of Proposition 3.4. For $0 \leq i \leq n$, the $s-1$ vertices between p_i and p_{i+2} along with p_i itself are labeled by $M_{(i,1)}, \dots, M_{(i,s)}$ in the direction from p_0 to p_{n+1} along the boundary of P .

Given a $\delta \in \mathcal{F}_{B_n}^s(k, 0)$, if δ contains no diameter, then there is a centrally symmetric subpolygon P' in the center of P . Fixing an r , we add to δ the diameter D in P' incident to a point of the form $q_{(a,r)}$, for some $a \geq n+2$, such that D connects $q_{(a,r)}$ to a point $M_{(i,j)}$ with the least i . Suppose that the diameter of δ is incident to the point $q_{(2n+1-b,m)}$ for some b, m ($0 \leq b \leq n-1$, $1 \leq m \leq s$). From this, the half subpolygon Q as well as the allocating labeling are determined as in the proof of Proposition 5.2. Then we turn δ into a full dissection δ' as follows. Note that δ dissects Q into $(sj+2)$ -gons, for some j . In each subpolygon Q' with $sd+2$ vertices ($d \geq 2$), locate the vertex $M_{(i,j)}$ with the least i such that $M_{(i,j)}$ could be connected to a vertex of the form $q_{(a,r)}$, for some $a \geq n+2$. Then we add $d-1$ diagonals incident to $M_{(i,j)}$. Note that when Q is fully dissected there are $n-k$ additional diagonals (including the diameter, if any), all of which have distinct initial points. Such a dissection is always possible for each r ($1 \leq r \leq s$) due to the periodicity of the universal labeling. These additional diagonals are colored *red*. By the map σ in Proposition 5.2, we construct the corresponding s -binomial path $\sigma(\delta')$ and color the peaks associated to red diagonals with a red east step. Note that the AMES of $\sigma(\delta')$ is at height $y = L_{(2n+1-b, m')}$, where $m' = m$ if b is even, and $m' = s-m+1$ otherwise. Moreover, the red peaks of $\sigma(\delta')$ are all at heights h such that $h \equiv r \pmod{s}$ if b is even, and $h \equiv s-r+1 \pmod{s}$ otherwise. The requested path $\zeta_1(\delta)$ is obtained from $\sigma(\delta')$ by converting the red peaks into diagonal steps. It follows that either $\zeta_1(\delta) \in \mathcal{B}_n^{(s,r)}(n-k, 0)^+$ or $\zeta_1(\delta) \in \mathcal{B}_n^{(s, s-r+1)}(n-k, 0)^-$.

To find ζ_1^{-1} , given a $\pi \in \mathcal{B}_n^{(s,r)}(n-k, 0)^+ \cup \mathcal{B}_n^{(s, s-r+1)}(n-k, 0)^-$, we turn π into an s -binomial path π' by converting the $n-k$ diagonal steps into peaks with a red east step. Then we locate the AMES of π' , say at height $y = L_{(2n+1-b, m')}$ for some b, m' ($0 \leq b \leq n-1$, $1 \leq m' \leq s$). From this, we determine the diameter of P , which is incident to the point

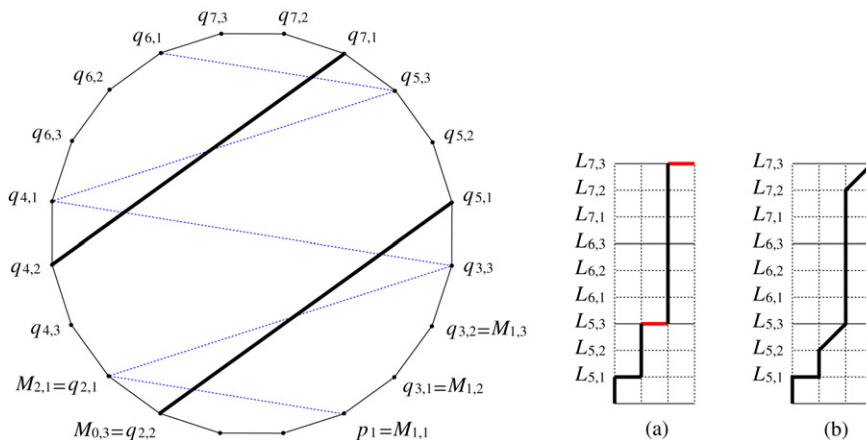


Fig. 11. A face of $\mathcal{F}_{B_3}^3(1, 0)$ and the corresponding path in $B_3^{(3,3)}(2, 0)^+$.

$q_{(2n+1-b,m)}$, where $m = m'$ if b is even and $m = s - m' + 1$ otherwise. Then the subpolygon Q as well as the allocating labeling are determined as in the proof of Proposition 5.2. By the map σ^{-1} in Proposition 5.2, we recover the full dissection $\sigma^{-1}(\pi') \in \mathcal{Q}_n^{(s)}(b, m)$. We associate the red peaks of π' to the diagonals in the dissection induced on Q as follows. If $\pi \in \mathcal{B}_n^{(s,r)}(n-k, 0)^+$ (respectively $\pi \in \mathcal{B}_n^{(s,s-r+1)}(n-k, 0)^-$), then for each red peak of π' , say at height $y = L_{(a,r)}$ (respectively $y = L_{(a,s-r+1)}$), we associate it to a diagonal in Q incident to the point $L_{(a,r)}$ (respectively $L_{(a,s-r+1)}$). If there is more than one diagonal incident to $L_{(a,r)}$ (respectively $L_{(a,s-r+1)}$), then choose the one connecting to the point $M_{(i,j)}$ with the least i . Such a diagonal is colored red. Then the subdivision on Q is obtained from $\sigma^{-1}(\pi')$ with the red diagonals removed. The corresponding dissection $\zeta_1^{-1}(\pi)$ is completely determined from Q .

The bijection ζ_2 can be established by the same construction. Since the argument works well for all r ($1 \leq r \leq s$), the assertion follows. \square

Example 5.7. On the left of Fig. 11 there is a face $\delta \in \mathcal{F}_{B_3}^3(1, 0)$, which consists of the B -diagonal $(q_{(7,1)}q_{(4,2)}, q_{(5,1)}q_{(2,2)})$. We try to find the corresponding path for the case $r = 3$. First, we add the diameter incident to the point $q_{(5,3)}$, connecting to the ‘least’ point $M_{(2,1)}$. Then the subpolygon Q as well as the allocating labeling are determined, as shown on the left of Fig. 9. Then we make Q fully dissected, denoted by δ' , by adding a diagonal incident to the point $q_{(3,3)}$. By the map σ in Proposition 5.2, we construct the path $\sigma(\delta')$ (see Example 5.3), shown in Fig. 11(a). Note that the red peaks of $\sigma(\delta')$ are at heights $y = L_{(5,3)}$ and $L_{(7,3)}$. The corresponding path $\zeta_1(\delta) \in B_3^{(3,3)}(2, 0)^+$ is obtained by converting the red peaks into diagonal steps, as shown in Fig. 11(b).

Example 5.8. On the left of Fig. 12 there is another face $\delta \in \mathcal{F}_{B_3}^3(1, 0)$, which consists of the B -diagonal $(q_{(7,1)}q_{(6,2)}, q_{(3,1)}q_{(2,2)})$. We try to determine the corresponding path for the case $r = 3$. First, we add the diameter incident to the point $q_{(6,3)}$, connecting to the ‘least’ point $M_{(1,3)}$. Then the subpolygon Q as well as the allocating labeling are determined, as shown on the left of Fig. 10. Then we make Q fully dissected, denoted by δ' , by adding a diagonal incident to the point $q_{(4,3)}$. By the map σ in Proposition 5.2, we construct the path $\sigma(\delta')$ (see Example 5.4), shown in

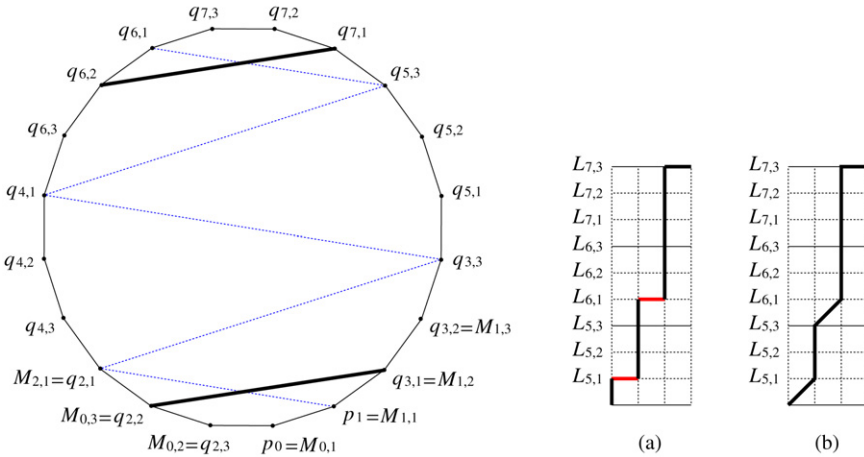


Fig. 12. A face of $\mathcal{F}_{B_3}^3(1, 0)$ and the corresponding path in $\mathcal{B}_3^{(3,1)}(2, 0)^-$.

Fig. 12(a). Note that the red peaks of $\sigma(\delta')$ are at heights $y = L_{(5,1)}, L_{(6,1)}$. The corresponding path $\zeta_1(\delta) \in \mathcal{B}_3^{(3,1)}(2, 0)^-$ is obtained by converting the red peaks into diagonal steps, as shown in Fig. 12(b).

5.4. For the faces of $\Delta^s(B_n)$ consisting of positive roots and negative roots

Consider the set $\mathcal{B}_n^{(s,r)}(d, t)$ of (s, r) -Delannoy paths with d diagonal steps, which visit the point $(t, 0)$ but not the point $(t + 1, 0)$. Given a $\pi \in \mathcal{B}_n^{(s,r)}(d, t)$, the canonical factorization of π is given by

$$\pi = E_1 \cdots E_t \mu_{t+1} \widehat{N}_t \mu_t \cdots \widehat{N}_1 \mu_1, \quad (13)$$

where $E_1 \cdots E_t$ are the initial t east steps, \widehat{N}_i is the last section of s consecutive north steps that go from the line $y = s(x - i)$ to the line $y = s(x - i + 1)$, for $1 \leq i \leq t$, each μ_i is a large (s, r) -Schröder path of a certain length, for $1 \leq i \leq t$, and $\mu_{t+1} \in \mathcal{B}_{n'}^{(s,r)}(d', 0)$ is an (s, r) -Delannoy path, for some $n', d' \geq 0$.

Example 5.9. On the right of Fig. 13 there is a path $\pi \in \mathcal{B}_5^{(3,3)}(3, 1)$ with 3 diagonal steps that visit the point $(1, 0)$ but not the point $(2, 0)$. The canonical factorization of π is $\pi = E \mu_2 \widehat{N} \mu_1$, where μ_2 is the section from B to C , \widehat{N} is the section from C to D , and μ_1 is the section from D to E . Note that \widehat{N} is the last section of three consecutive north steps that go from the line $y = s(x - 1)$ to the line $y = sx$, and that $\mu_2 \in \mathcal{B}_3^{(3,3)}(2, 0)$, as shown in Example 5.7.

By the canonical factorization (13) of $\pi \in \mathcal{B}_n^{(s,r)}(d, t)$, we know that either μ_{t+1} is empty or $\mu_{t+1} \in \mathcal{B}_{n'}^{(s,r)}(d', 0)$, for some $n' \geq 1$, and that $\mathcal{B}_{n'}^{(s,r)}(d', 0) = \mathcal{B}_{n'}^{(s,r)}(d', 0)^+ \cup \mathcal{B}_{n'}^{(s,r)}(d', 0)^-$, as defined in (11) and (12). (Note that $\mathcal{B}_{n'}^{(s,r)}(d', 0)^- = \emptyset$ if $n' = 1$.) We partition the set $\mathcal{B}_n^{(s,r)}(d, t)$ into two subsets $\mathcal{B}_n^{(s,r)}(d, t)^+$ and $\mathcal{B}_n^{(s,r)}(d, t)^-$ according to the canonical factorization such that $\pi \in \mathcal{B}_n^{(s,r)}(d, t)^+$ if either μ_{t+1} is empty or $\mu_{t+1} \in \mathcal{B}_{n'}^{(s,r)}(d', 0)^+$ for some $n' \geq 1$, and

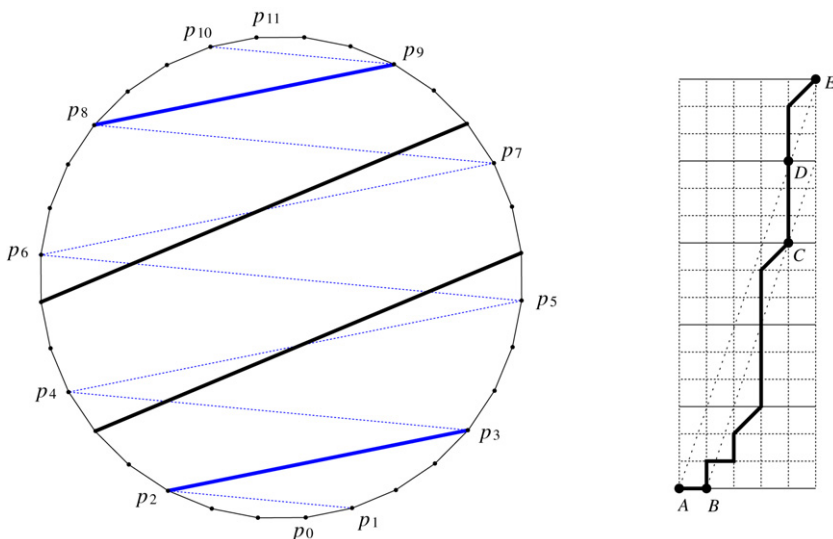


Fig. 13. A face of $\mathcal{F}_{B_5}^3(1, 1)$ and the corresponding path in $\mathcal{B}_5^{(3,3)}(3, 1)^+$.

$\pi \in \mathcal{B}_n^{(s,r)}(d, t)^-$ otherwise. Making use of the bijections in Propositions 3.4 and 5.6 as building blocks, we establish the following pair of bijections, as requested in Theorem 5.1.

Proposition 5.10. For $0 \leq k \leq n$ and for each r ($1 \leq r \leq s$), there is a bijection ξ_1 between the set $\mathcal{F}_{B_n}^s(k, t)$ and the set $\mathcal{B}_n^{(s,r)}(n-k-t, t)^+ \cup \mathcal{B}_n^{(s,s-r+1)}(n-k-t, t)^-$, and there is a bijection ξ_2 between the set $\mathcal{F}_{B_n}^s(k, t)$ and the set $\mathcal{B}_n^{(s,r)}(n-k-t, t)^- \cup \mathcal{B}_n^{(s,s-r+1)}(n-k-t, t)^+$.

Proof. The case $t = 0$ is given in Proposition 5.6. Assume that $t \geq 1$. Given a $\delta \in \mathcal{F}_{B_n}^s(k, t)$, let the negative simple roots of δ be $-\alpha_{j_1}, \dots, -\alpha_{j_t}$, where $1 \leq j_1 < \dots < j_t \leq n$. The polygon P is dissected by these roots into a centrally symmetric subpolygon, denoted by Q_{t+1} , in the center of P and t subpolygons on either side of Q_{t+1} . Let Q_1, \dots, Q_t be the t subpolygons on one side of Q_{t+1} , where Q_i and Q_{i+1} touch each other along the diagonal $-\beta_{j_i} = p_{j_i} p_{j_i+1}$ ($1 \leq i \leq t$). Note that Q_{t+1} has $2sn_{t+1} + 2$ vertices, where $n_{t+1} = n - j_t$, which degenerates to the diameter $p_n p_{n+1}$ if $j_t = n$. Each Q_i has $sn_i + 2$ vertices, where $n_i = j_i - j_{i-1}$ (we assume that $j_0 = 0$). Let δ_i be the subdivision induced on Q_i . Suppose that δ_i consists of k_i diagonals, so that $k_1 + \dots + k_{t+1} = k$.

Fix an r . In Q_{t+1} , by the map ζ_1 in Proposition 5.6, we construct the corresponding path $\omega_{t+1} = \zeta_1(\delta_{t+1})$. If $\omega_{t+1} \in \mathcal{B}_{n_{t+1}}^{(s,r)}(n_{t+1} - k_{t+1}, 0)^+$, then we associate each δ_i ($1 \leq i \leq t$) with the corresponding path $\varphi(\delta_i) \in \mathcal{A}_{n_i-1}^{(s,r)}$ (with a single block) by the map φ in Proposition 3.4. Otherwise $\omega_{t+1} \in \mathcal{B}_{n_{t+1}}^{(s,s-r+1)}(n_{t+1} - k_{t+1}, 0)^-$, and we associate δ_i with the corresponding path $\varphi(\delta_i) \in \mathcal{A}_{n_i-1}^{(s,s-r+1)}$. For $1 \leq i \leq t$, we factorize $\varphi(\delta_i)$ as $\varphi(\delta_i) = \widehat{N}\omega_i E$. Then the requested path $\xi_1(\delta)$ is defined by

$$\xi_1(\delta) = E_1 \cdots E_t \omega_{t+1} \widehat{N}\omega_t \cdots \widehat{N}\omega_1.$$

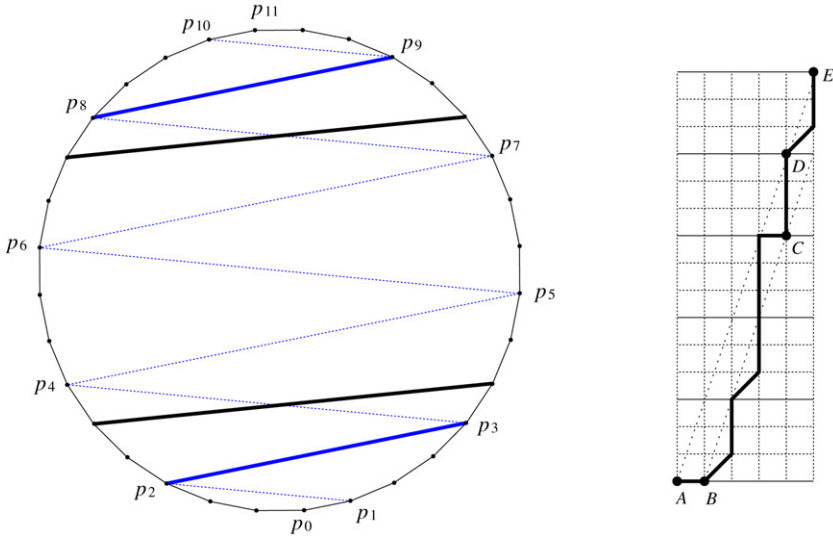


Fig. 14. A face of $\mathcal{F}_{B_5^3}(1, 1)$ and the corresponding path in $\mathcal{B}_5^{(3,1)}(3, 1)^-$.

We observe that the length of $\xi_1(\delta)$ is $t + n_{t+1} + \sum_{i=1}^t (n_i - 1) = n$, and the number of diagonal steps in $\xi_1(\delta)$ is $(n_{t+1} - k_{t+1}) + \sum_{i=1}^t (n_i - 1 - k_i) = n - k - t$. Hence either $\xi_1(\delta) \in \mathcal{B}_n^{(s,r)}(n - k - t, t)^+$ or $\xi_1(\delta) \in \mathcal{B}_n^{(s,s-r+1)}(n - k - t, t)^-$.

To find ξ_1^{-1} , given a $\pi \in \mathcal{B}_n^{(s,r)}(n - k - t, t)^+ \cup \mathcal{B}_n^{(s,s-r+1)}(n - k - t, t)^-$, we factorize π into the canonical factorization $\pi = E_1 \cdots E_t \mu_{t+1} \hat{N}_t \mu_t \cdots \hat{N}_1 \mu_1$. Suppose that μ_i is of length n'_i with d_i diagonal steps, so that $n'_1 + \cdots + n'_{t+1} = n - t$ and $d_1 + \cdots + d_{t+1} = n - k - t$. For $1 \leq i \leq t$, let $j_i = n'_1 + \cdots + n'_i + i$. To recover the dissection $\xi_1^{-1}(\pi)$, we add the t negative simple roots $-\alpha_{j_1}, \dots, -\alpha_{j_t}$ in the s -snake, which specify the subpolygons Q_1, \dots, Q_{t+1} accordingly, and then recover the subdivision of Q_i from μ_i by the maps in Propositions 3.4 and 5.6. Since Q_i contains $n'_i - d_i$ diagonals ($1 \leq i \leq t + 1$), $\xi_1^{-1}(\pi)$ contains $\sum_{i=1}^{t+1} (n'_i - d_i) = k$ positive roots. Hence $\xi_1^{-1}(\pi) \in \mathcal{F}_{B_n^s}(k, t)$.

Making use of the map ζ_2 in Proposition 5.6, the bijection ξ_2 can be established by the same construction. Since the argument works well for all r ($1 \leq r \leq s$), the proof is completed. \square

Example 5.11. On the left of Fig. 13 there is a face $\delta \in \mathcal{F}_{B_5^3}(1, 1)$, which contains the negative simple root $-\alpha_2 = (p_2 p_3, p_8 p_9)$. We try to determine the corresponding path for the case $r = 3$. The polygon P is dissected into a centrally symmetric subpolygon Q_2 and an octagon Q_1 on one side of Q_2 , where Q_2 (respectively Q_1) contains the section $p_3 \cdots p_8$ (respectively $p_1 p_2$) of the s -snake. The dissection induced on Q_2 is carried to a path $\omega_2 \in \mathcal{B}_3^{(3,3)}(2, 0)^+$ (see Example 5.7), shown in Fig. 11(b). So we associate the dissection induced on Q_1 to a $(3, 3)$ -Schröder path $\hat{N}\omega_1 E$ of length 2 by the map in Proposition 3.4. The corresponding path $\xi_1(\delta) = E\omega_2 \hat{N}\omega_1 \in \mathcal{B}_5^{(3,3)}(3, 1)^+$ is shown on the right of Fig. 13, where ω_2 is the section from B to C and ω_1 is the section from D to E .

On the left of Fig. 14 there is another face $\delta' \in \mathcal{F}_{B_5^3}(1, 1)$. The dissection induced on Q_2 is carried to a path $\omega'_2 \in \mathcal{B}_3^{(3,1)}(2, 0)^-$ (see Example 5.8), shown in Fig. 12(b). So we associate the dissection induced on Q_1 to a $(3, 1)$ -Schröder path $\hat{N}\omega'_1 E$ of length 2 by the map in Propo-

sition 3.4. The corresponding path $\xi_1(\delta') = E\omega'_2\widehat{N}\omega'_1 \in \mathcal{B}_5^{(3,1)}(3, 1)^-$ is shown on the right of Fig. 14.

6. Concluding remarks

In Fuss–Catalan combinatorics, a main theme is the still-sought explanation of the mysterious equinumerology “CC=NN=NC” of three structures, where CC stands for the generalized cluster complex, NN the generalized nonnesting partitions, and NC the generalized noncrossing partitions [1]. In this paper we propose a fourth model in the language of lattice paths of types *A* and *B* and establish their correspondences with the generalized cluster complexes. We hope that this approach not only offers a new combinatorial model interesting in its own right but also helps enrich this wonderful topic.

A natural question is to find the type-*D* lattice path model, in connection to $\Delta^s(D_n)$. For example, when $s = 1$ it is not hard to design a lattice path model with the total number of paths given by $\frac{3n-2}{n} \binom{2n-2}{n-1}$. However, as suggested in this work, a ‘correct’ model should be consistent with the entries of the *F*-triangle when taking refinement according to some statistic as in types *A* and *B*. This task seems to be quite daunting since the formula for the *F*-triangle of type *D* is complicated. Namely, for $n \geq 2$ and $s \geq 1$, the face number $|\mathcal{F}_{D_n}^s(k, t)|$ of $\Delta^s(D_n)$ with k positive roots and t negative roots is given by $\binom{n}{k+t} \binom{s(n-1)+k-1}{k} + s \binom{n-1}{k+t-1} \binom{s(n-1)+k-2}{k-1} - \delta_{t,0} \frac{1}{n-1} \binom{n-1}{k-1} \binom{s(n-1)+k-1}{k}$ [13]. So far we are not able to find such a model, and we are very interested in it.

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